OBSERVERS FOR HYBRID SYSTEMS WITH CONTINUOUS STATE RESETS

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Abstract

A methodology for the design of dynamical observers for hybrid plants has been recently proposed in [3]. The hybrid observer consists of two parts: a *location observer* that identifies the current location of the hybrid plant and a *continuous observer* that estimates the continuous state of the hybrid plant. When an appropriate set of properties on the hybrid plant is satisfied, the hybrid observer identifies the current location of the plant after a finite number of steps and converges exponentially to the continuous state. In this note the previous result is extended to hybrid models with continuous state resets.

1 Introduction

The authors investigated for years the use of a hybrid formalism to solve control problems in automotive applications (see [1]). The hybrid control algorithms they developed are usually based on full state feedback, while only partial information about the state of the hybrid plant is often available. This motivates the work on the design of observers for hybrid systems. Some partial results are given in [2], where an application to a power-train control problem is considered.
In their previous paper [3], they proposed a procedure for the design of hybrid observers that reconstruct the complete hybrid state from the knowledge of the hybrid plant inputs and outputs, achieving correct identification of the plant location sequence and exponential convergence of the continuous state estimation error. The hybrid observer consists of two parts: a *location observer* and a *continuous observer*. The former identifies the current location of the hybrid plant, while the latter produces an estimate of the evolution of the continuous state of the hybrid plant. In [3], two cases are studied: in the first case, the current discrete state of the given hybrid plant can be reconstructed using the discrete input/output information only, without the need of additional information from the evolution of the continuous part of the plant. Then, on the basis of this information, an estimate of the plant continuous state is constructed using the continuous plant input/output. Secondly, the case in which the evolutions of the discrete inputs and outputs of the hybrid plant are not sufficient to estimate the current location is tackled. In this case, the continuous plant inputs and outputs are used to obtain some additional information useful for the identification of the current plant discrete state.
In this note the results presented in [3] are extended to hybrid plant models that contain continuous state resets.
The paper is organized as follows: in section 2

the general scheme of the hybrid observer is reported and in sections 3 the synthesis of hybrid observers for hybrid plant models that allow continuous state resets is presented.

2 Problem formulation

In this paper we consider the class of hybrid systems with linear continuous–time dynamics. A hybrid system $\mathcal{H}$ of this class can be described by a tuple

$$\mathcal{H} = (Q, \Sigma, \Psi, \varphi, \phi, \eta, X, U, Y, f, h, r)$$

where $Q = \{q_1, \cdots, q_N\}$ is the finite set of discrete states (locations) with $N = |Q|$, Σ is the finite set of possible input and internal events, Ψ is the finite set of discrete outputs, and $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$, and $Y \subseteq \mathbb{R}^p$ are the continuous state, control and output domains, respectively. The functions φ, ϕ and η characterize the dynamics of the discrete variables of the system as follows:

$$\begin{aligned} q(k+1) &\in \varphi\left(q(k), \sigma(k+1)\right) & (1)\\ \sigma(k+1) &\in \phi\left(q(k), x(t_{k+1}^-), u(t_{k+1}^-)\right) & (2)\\ \psi(k+1) &\in \eta\left(q(k), \sigma(k+1), q(k+1)\right) & (3) \end{aligned}$$

where $q(k) \in Q$ and $\psi(k) \in \Psi$ are, respectively, the location and the discrete output after the k-th input event $\sigma(k) \in \Sigma \bigcup \{\epsilon\}$, and t_k denotes the unknown time at which this event takes place[1].

The set–valued functions $\varphi : Q \times \Sigma \to 2^Q$ and $\eta : Q \times \Sigma \times Q \to \Psi$ are the transition and output functions respectively. The function $\phi : Q \times X \times U \to 2^\Sigma$ specifies the subset of events that can be executed at the current time, for a given location and given values of the continuous state $x(t) \in X$ and continuous input $u(t) \in U$ of the system. It is worth noting that the finite set Σ is composed by both exogenous input events and internal events depending on the continuous state x and input u.

The function $f : Q \times X \times U \to \mathbb{R}^n$ and $h : Q \times X \to Y$ define the dynamics of the continuous variables of the hybrid systems. They are assumed to be linear and time-invariant and described as follows:

$$\begin{aligned} \dot{x}(t) &= f(q_i, x(t), u(t)) = A_i x(t) + B_i u(t) & (4)\\ y(t) &= h(q_i, x(t)) = C_i x(t) & (5) \end{aligned}$$

where $y(t) \in Y$ is the continuous output of the system and $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$ depend on the current plant location q_i.

Finally, the function $r : Q \times Q \times X \to X$ describes the continuous state resets associated to the hybrid system transitions. For each transition $q_i \to q_j$ the reset function is assumed to be affine and described by

$$x(t_k) = r(q_i, q_j, x(t_k^-)) = R_{ij}^1\, x(t_k^-) + R_{ij}^0 \qquad (6)$$

where t_k denotes the transition time and $R_{ij}^1 \in \mathbb{R}^{n\times n}$, $R_{ij}^0 \in \mathbb{R}^n$.

In this paper we consider the problem of designing a hybrid observer for hybrid plants described as illustrated above. We assume that there is a minimum separation time D between any two consecutive switchings of the hybrid plant. This time interval is usually referred to as the dwell time (see [6]).

The hybrid observer $\mathcal{H}_O$ is a hybrid system itself and its task is to provide an estimate $\tilde{q}(k)$ and an estimate $\tilde{x}(t)$ for the current location $q(k)$ and continuous state $x(t)$ of the hybrid plant, respectively. Its inputs are the plant continuous input and output, u and y, and the plant discrete output ψ. We denote by $\hat{t}_k$ the time at which the hybrid observer takes the k-th transition.

In particular, we present a methodology for the design of *exponentially ultimately bounded hybrid observers* defined as follows:

[1] The event ϵ is the *silent event* and it is introduced to model different possible situations for the discrete dynamics. For example, if $\phi(q, x, u) = \{\epsilon\}$, then there is no discrete transition enabled for the given values of x and u while if $\phi(q, x, u) = \{\overline{\sigma}, \epsilon\}$, then it is possible either to let time pass or to take the discrete transition associated to $\overline{\sigma}$. Moreover, if $\phi(q, x, u) = \{\overline{\sigma}\}$, then the discrete transition associated to $\overline{\sigma}$ is forced to occur. This is useful for example to model internal transitions due to the continuous state hitting a guard. We assume that there are no infinitely fast event sequences.

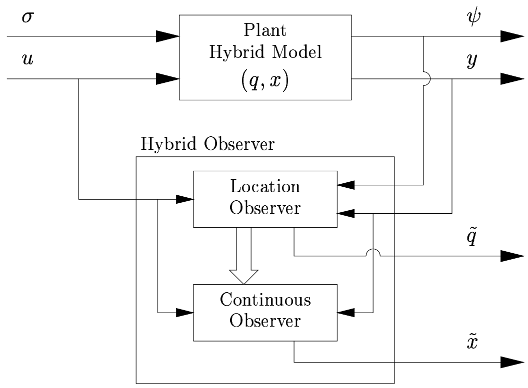

Figure 1: Observer structure: location observer and continuous observer.

Definition 1 *Given a hybrid system $\mathcal{H}$ as in (1–6), a hybrid observer $\mathcal{H}_O$ is said to be* exponentially ultimately bounded *if there exists a positive integer K and constants $c \geq 1$, $\mu > 0$ and $b \geq 0$ such that, $\forall k \geq K$ and $\forall t > \hat{t}_K$,*

$$\tilde{q}(k) = q(k)$$
$$\|\tilde{x}(t) - x(t)\| \leq c e^{-\mu(t-\hat{t}_K)} \|\tilde{x}(\hat{t}_K) - x(t_K)\| + b$$

for every initial hybrid state $(q(0), x(0)) \in Q \times X$, every continuous input $u(\tau)$ with $\tau \in [0, t]$, every possible input sequence $\sigma(1), \ldots, \sigma(k)$ and every feasible output sequence $\psi(1), \ldots, \psi(k)$. The constant μ is the rate of convergence and b is the ultimate bound.

Given a hybrid plant with state (q, x), the structure of the proposed hybrid observer $\mathcal{H}_O$ is illustrated in Figure 1. The *location observer* describes the evolution of the discrete location of $\mathcal{H}_O$ while the *continuous observer* governs the evolution of the continuous state of $\mathcal{H}_O$.

The *location observer* receives as input the continuous plant input $u(t)$ and hybrid output $(\psi(k), y(t))$. Its task is to provide the estimate $\tilde{q}(k)$ of the discrete location $q(k)$ of the hybrid plant at the current time. Based on the discrete evolution of the location observer, the *continuous observer* constructs an estimate $\tilde{x}(t)$ of the plant continuous state that converges exponentially to $x(t)$. The continuous plant input $u(t)$ and output $y(t)$ are used by the continuous observer to this purpose.

3 Hybrid Observer Design

In this section, exploiting the result presented in [3], sufficient conditions for the design of a location observer and a continuous observer achieving exponential ultimate boundness according to Definition 1 are given.

3.1 Location-observer design

Let us introduce a definition that will be used in the sequel:

Definition 2 *Let us denote by $\mathcal{D}_{\mathcal{H}}$ the discrete event system (DES) associated to the hybrid system $\mathcal{H}$ defined by:(1),*

$$\sigma(k+1) \in \widehat{\phi}\left(q(k)\right) = \bigcup_{x \in X, u \in U} \phi\left(q(k), x, u\right)$$

and (3). The hybrid system $\mathcal{H}$ is said to be current–location observable *if the DES $\mathcal{D}_{\mathcal{H}}$ is alive and there exists a positive integer K such that for every $i \geq K$ and any unknown initial location $q_0 \in Q$, the state $q(i)$ of $\mathcal{D}_{\mathcal{H}}$ can be determined from the output sequence $\psi(1), \ldots, \psi(i)$ for every possible input sequence $\sigma(1), \ldots, \sigma(i)$.*

A straightforward design of an observer producing estimates of the state $q(k)$ after each output $\psi(k)$ can be done by computing the subset $\tilde{q}(k)$ of possible states $q(k)$ that the DES $\mathcal{D}_{\mathcal{H}}$ could have entered when the last event $\sigma(k)$ occurred. The observer $\mathcal{O}$ is a discrete event system itself

$$\mathcal{O} = (Q_{\mathcal{O}}, \Sigma_{\mathcal{O}}, \Psi_{\mathcal{O}}, \varphi_{\mathcal{O}}, \phi_{\mathcal{O}}, \eta_{\mathcal{O}})$$

where $Q_{\mathcal{O}} \subseteq 2^Q$, $\Sigma_{\mathcal{O}} = \Psi$, $\Psi_{\mathcal{O}} = Q_{\mathcal{O}}$, and $\eta_{\mathcal{O}} = \varphi_{\mathcal{O}}$. Its dynamics are described by

$$\tilde{q}(k+1) = \varphi_{\mathcal{O}}\left(\tilde{q}(k), \psi(k+1)\right) \quad (7)$$
$$\tilde{\sigma}(k+1) \in \phi_{\mathcal{O}}\left(\tilde{q}(k)\right) \quad (8)$$
$$\tilde{\psi}(k+1) = \varphi_{\mathcal{O}}\left(\tilde{q}(k), \psi(k+1)\right) = \tilde{q}(k+1) \quad (9)$$

where the input of the observer is the output $\psi(k)$ of $\mathcal{D}_{\mathcal{H}}$ and $\tilde{q}(k) = \tilde{\psi}(k) \in Q_{\mathcal{O}}$ is the observer state (and output) and correspond to the subset of possible states $q(k)$ that the system $\mathcal{D}_{\mathcal{H}}$ entered when the last output $\psi(k)$ was observed.

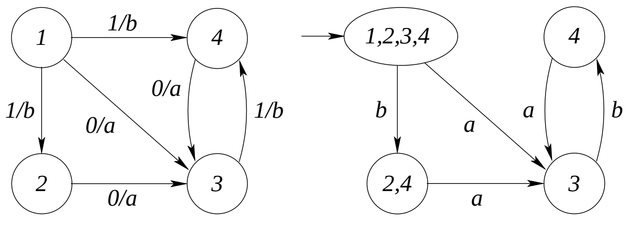

Figure 2: DES $\mathcal{D}_1$ (left) and observer $\mathcal{O}_1$ (right).

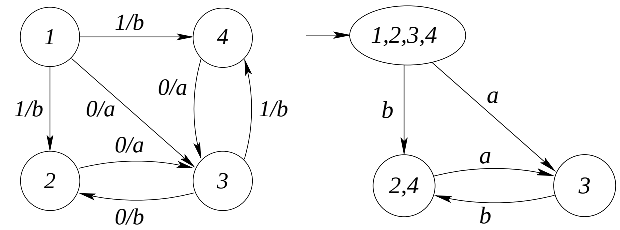

Figure 3: DES $\mathcal{D}_2$ (left) and observer $\mathcal{O}_2$ (right).

The observer transition function $\varphi_{\mathcal{O}}$ and the admissible event function $\phi_{\mathcal{O}}$ can be constructed by inspection of the given discrete event system $\mathcal{D}_{\mathcal{H}}$ following the algorithm for the computation of the *current-state observation tree* as described in [4]. The construction starts from the initial state $\tilde{q}(0)$: since the initial state of $\mathcal{D}_{\mathcal{H}}$ is unknown, then $\tilde{q}(0) = Q$. When the first input event $\psi(1)$ is received, the observer makes a transition to the state $\tilde{q}$ corresponding to the set

$$\tilde{q} = \{q \in Q \mid \exists s \in \tilde{q}(0) \text{ and } \sigma \in \phi(s) : \\ q \in \varphi(s,\sigma), \text{ with } \psi(1) \in \eta(s,\sigma,q)\}$$

that depends on the value of $\psi(1)$. In fact, the number of observer states at the second level depends on the number of possible distinct events $\psi(1)$. By iterating this step, one can easily construct the third level of the tree whose nodes correspond to the sets of possible states that $\mathcal{D}_{\mathcal{H}}$ entered after the second event. Since this procedure produces at most $2^N - 1$ observer states, then the construction of the observer necessarily ends.

Consider for example the DES $\mathcal{D}_1$ in figure 2 for which $Q = \{1,2,3,4\}$, $\Sigma = \{0,1\}$ and $\Psi = \{a,b\}$. The observer $\mathcal{O}_1$ of this DES has four states, i.e. $Q_{\mathcal{O}} = \{\{1,2,3,4\},\{2,4\},\{3\},\{4\}\}$ (see figure 2).

The following theorem gives necessary and sufficient conditions for a hybrid system $\mathcal{H}$ to be current–location observable in terms of the observer $\mathcal{O}$ obtained for the DES $\mathcal{D}_{\mathcal{H}}$ associated to $\mathcal{H}$. The theorem has its origins in a result of [9], where a different definition of observability was considered.

Theorem 3 *A hybrid system $\mathcal{H}$, with alive associated DES $\mathcal{D}_{\mathcal{H}}$, is current–location observable if and only if the observer $\mathcal{O}$ of $\mathcal{D}_{\mathcal{H}}$ defined as in (7–9) verifies:*
(i) the subset $E_{\mathcal{O}} \subseteq Q_{\mathcal{O}}$ of singleton states of $\mathcal{O}$ is nonempty;
(ii) every primary cycle of $\mathcal{O}$ includes at least one state in $E_{\mathcal{O}}$;
(iii) the subset $E_{\mathcal{O}}$ is $\varphi_{\mathcal{O}}$-invariant.[2]

The following examples illustrate how Theorem 3 works. For the DES $\mathcal{D}_1$ and the corresponding observer $\mathcal{O}_1$ in figure 2, $E_{\mathcal{O}} = \{\{3\},\{4\}\}$ and the only cycle of $\mathcal{O}_1$ is composed of states of $E_{\mathcal{O}}$. Moreover it is easy to verify that the set $E_{\mathcal{O}}$ is invariant. Then, $\mathcal{D}_1$ is current–location observable. Consider next the DES $\mathcal{D}_2$ and its observer $\mathcal{O}_2$ in Figure 3. The observer has three states: $Q_{\mathcal{O}} = \{\{1,2,3,4\},\{2,4\},\{3\}\}$, $E_{\mathcal{O}} = \{\{3\}\}$ and the only cycle of $\mathcal{O}_2$ includes state 3. However, since state $E_{\mathcal{O}}$ is not $\varphi_{\mathcal{O}}$–invariant, then $\mathcal{D}_2$ is not current–location observable. Note that while it is easy to check whether conditions *(i)* and *(ii)* of Theorem 3 are satisfied, verifying condition *(iii)* is more involved. An algorithm of complexity $O(|Q_{\mathcal{O}}|)$ to check $\varphi_{\mathcal{O}}$-invariance can be found in [10].

When the evolutions of the discrete inputs and outputs of the hybrid plant are not sufficient to estimate the current location, the continuous plant inputs and outputs can be used to obtain some additional information. It is worth noting that unfortunately, the processing of the continuous signals of the plant gives reliable discrete information only after some delay with respect to plant location switchings and this results in a coupling between the location observer parameters and the continuous observer parameters.

[2] Following [10], a set S is said to be φ-invariant if $\bigcup_{q\in S}\bigcup_{\sigma\in\phi(q)}\varphi(q,\sigma) \subset S$.

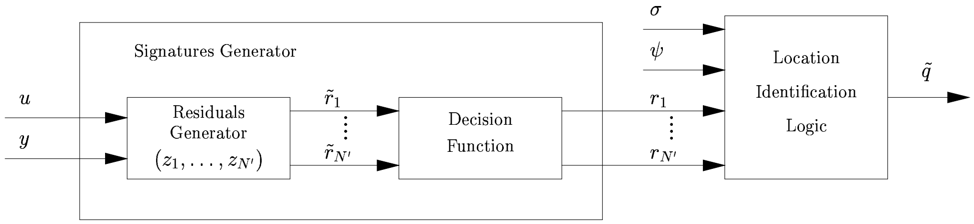

Figure 4: Location observer structure.

Residual signals can be used to detect a change in the continuous dynamics of the plant and the resulting signatures can be used as additional inputs to the current–location observer.

Introduction of signatures. Consider for example a hybrid plant whose discrete behavior is represented by the system $\mathcal{D}_2$ depicted in Figure 3. Assume that a signature r_2 can be produced to detect the continuous dynamics associated to location 2 (which is assumed to be different from the others) as soon as the hybrid plant enters location 2. Then, signal r_2 can be used as an input for the discrete observer. A representation of the DES associated to the hybrid plant composed with the generator of the signature r_2 can be obtained by adding an output r_2 to each arc entering location 2. By doing this the DES $\mathcal{D}_3$ shown in Figure 5 is obtained from $\mathcal{D}_2$. By the introduction of signature r_2, the hybrid system becomes current–location observable. Figure 5 shows the observer of $\mathcal{D}_3$, obtained applying the synthesis described in (7–9).

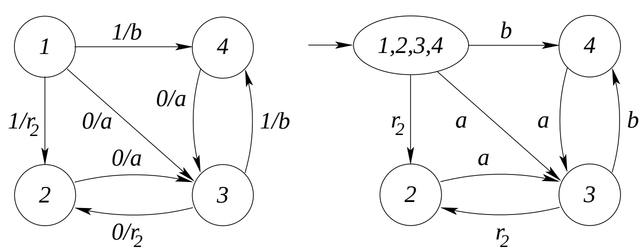

Figure 5: DES $\mathcal{D}_3$ (left) and observer $\mathcal{O}_3$ (right).

In the general case, if the hybrid plant is not current–location observable, then one may introduce a number of signatures detecting some of the different continuous dynamics of the plant to achieve current–location observability for the combination of the hybrid plant and the signature generator. Necessary and sufficient conditions for current–location observability of the composition hybrid plant and the signature generator are given in Theorem 3. If dynamics parameters in (4–5) are different in each location, then current–location observability can always be achieved in this way. The complete scheme of the location observer is shown in Figure 4. The *signatures generator* is described in the following section and the *location identification logic* is the discrete observer (7–9).

Signatures generator. The task of the signature generator is similar to that of a fault detection and identification algorithm (see [8] for a tutorial). Indeed, the signatures generator has to decide whether or not the continuous system is obeying to a particular dynamics in a set of known ones. Assuming that the location observer has properly recognized that the hybrid plant is in location q_i, i.e. $\tilde{q} = \{q_i\}$, then the location observer should detect a fault from the evolution of $u(t)$ and $y(t)$ when the hybrid plant changes the location to some $q_j \neq q_i$ and should identify the new location q_j. The time delay in the location change detection and identification is critical to the convergence of the overall hybrid observer. We denote by Δ an upper bound for such delay and we assume that it is lower than the plant dwell time D (see [6]), i.e. $\Delta < D$.

Since, when a change of location occurs, the continuous dynamics of the plant suddenly change, then the fault detection algorithms of interest are those designed for abrupt faults [5]. The general scheme is composed of three cascade blocks: the *residuals generator*, the *decision function*, and the fault decision logic, renamed here *location identification logic*, see Figure 4. The *signature generator* is the pair residuals generator–decision function. Assume that, in order to achieve current–

location observability for the hybrid plant the signature generator has to detect N' different continuous dynamics (4–5) associated to a subset of states $\mathcal{R} \subseteq Q$. The simplest and most reliable approach for our application is to use a bank of N' Luenberger observers (see [5]), one for each plant dynamics in $\mathcal{R}$, as residual generators:

$$\dot{z}_j(t) = H_j z_j(t) + B_j u(t) + L_j y(t) \quad (10)$$
$$\tilde{r}_j(t) = C_j z_j(t) - y(t) \quad (11)$$

where $H_j = A_j - L_j C_j$ and L_j are design parameters. The N' residual signals $\tilde{r}_j$ are used to identify the continuous dynamics the plant is obeying to. Indeed, no–vanishing residuals $\tilde{r}_j(t)$ correspond to $j \neq i$. The decision function outputs N' binary signals as follows:

$$r_j(t) = \begin{cases} \textit{true} & \text{if } \|\tilde{r}_j(t)\| \leq \varepsilon \\ \textit{false} & \text{if } \|\tilde{r}_j(t)\| > \varepsilon \end{cases} \quad (12)$$

for $j = 1, \dots, N'$, where the threshold ε is a design parameter. In the following theorem, a sufficient condition for ensuring $r_i = \textit{true}$ within a time Δ after a transition of the hybrid plant to a dynamics (A_i, B_i, C_i) is presented.

Theorem 4 *For a given $\Delta > 0$, $\varepsilon > 0$ and a given upper bound Z_0 on $\|x - z_i\|$, if the estimator gains L_i in (10) are chosen such that $H_i = A_i - L_i C_i$ have distint eigenvalues and*

$$\alpha(H_i) \leq -\frac{1}{\Delta} \log \frac{\|C_i\| \, k(H_i) \, Z_0}{\varepsilon} \quad (13)$$

then r_i becomes true before a time Δ elapses after a change in the plant dynamics parameters to the values (A_i, B_i, C_i).

Consider the j–th residual generator and assume that there is a transition from location q_j to location q_i, so that the continuous state x of the hybrid plant is governed by dynamics defined by parameters $(A_i, B_i, C_i) \neq (A_j, B_j, C_j)$. Unfortunately, as shown for example by the following theorem, there are cases where we cannot prevent the signal r_j from remaining *true* for an unbounded time:

Theorem 5 *If the matrix $(C_j - C_i)B_i + C_j(B_i - B_j)$ is invertible, with $i \neq j$, then for any hybrid plant initial condition, the class of plant inputs $u(t)$ that achieves $r_j(t) = true$ for all $t > \Delta$ after a change in the plant dynamics parameters to (A_i, B_i, C_i) is not empty.*

In the general case, the set of configurations and the class of plant inputs for which the signatures (12) fail to properly identify the continuous dynamics can be obtained by computing the maximal safe set and the maximal controller for dynamics (4–5) and (10-11) with respect to a safety specification defined in an extended state space that contains an extra variable τ representing the elapsed time after a plant transition. More precisely, the set of configurations for which a wrong signature may be produced up to a time $t' > \Delta$ after a plant location change, is given by those configurations $(0, x^0, z_j^0)$ from which there exists a plant continuous input $u(t)$ able to keep the trajectory inside the set $[0, t'] \times \left\{(x, z_j) \in \mathbb{R}^{2n} | \; \| C_j \, z_j - C_i \, x \| \leq \varepsilon \right\}$. However, since in pratical applications the resulting maximal controller is very small, the case of not proper identification is unlikely to occur.

3.2 Continuous observer design

The continuous observer, which[3] describes the evolution of the estimate $\tilde{x}(t)$ of the plant continuous state $x(t)$, is as follows:

1. to each location observer state $\tilde{q}$ is associated the continuous dynamics

$$\begin{cases} \dot{\tilde{x}}(t) = 0 & \text{if } \tilde{q} \in Q_{\mathcal{O}} \setminus E_{\mathcal{O}} \\ \dot{\tilde{x}}(t) = (A_i - G_i C_i)\, \tilde{x}(t) + B_i u(t) + G_i y(t) & \\ & \text{if } \tilde{q} = \{q_i\} \in E_{\mathcal{O}} \end{cases} \quad (14)$$

   where A_i, B_i, C_i are as in (4–5), and the observer gain matrix $G_i \in \mathbb{R}^{n \times p}$ is the design parameter used to set the velocity of convergence in each location $\tilde{q} \in E_{\mathcal{O}}$.

2. to each location observer transition $\{q_i\} \to \{q_j\}$, with $\{q_i\} \in E_{\mathcal{O}}$ and $q_j \in \text{Reach}(q_i) =$

$\left\{q \mid q \in \varphi(q_i, \sigma) \text{ with } \sigma \in \bar{\phi}(q_i)\right\}$, is associated the reset

$$\tilde{x}(\hat{t}_k) = \tilde{x}(\hat{t}_k^+) = R_{ij}^1 \, \tilde{x}(\hat{t}_k^-) + R_{ij}^0 \qquad (15)$$

where $\hat{t}_k$ denotes the k-th location observation transition time, and R_{ij}^1, R_{ij}^0 are as in (6).

Exponential convergence of the continuous observer is analyzed considering the complete hybrid system $\mathcal{H}_P \otimes \mathcal{H}_O$ obtained by composing the hybrid model $\mathcal{H}_P$ and the observer hybrid model $\mathcal{H}_O$ as defined above, from the time $\hat{t}_K$ at which the location observer enters the $\varphi_{\mathcal{O}}$-invariant subset $E_{\mathcal{O}}$ of singleton locations.

The discrete states of the overall hybrid system $\mathcal{H}_P \otimes \mathcal{H}_O$ are of type $(q_i, \{q_j\})$, the former corresponding to plant locations $q_i \in Q$ and the latter corresponding to observer locations $\{q_j\} \in Q_{\mathcal{O}}$. The continuous dynamics of the overall hybrid system $\mathcal{H}_P \otimes \mathcal{H}_O$ that govern the evolution of the composed state (x, ζ) for locations in the subset $Q \times E_{\mathcal{O}}$ are as follows:

1. to each location $(q, \tilde{q})$ in the subset $Q \times E_{\mathcal{O}}$ are associated the continuous dynamics

$$\dot{x}(t) \;=\; A_i \, x(t) + B_i \, u(t) \quad \text{if } q = q_i \qquad (16)$$

and

$$\begin{cases} \dot{\zeta}(t) = F_i \, \zeta(t) & \text{if } \tilde{q} = \{q_i\} \\ \dot{\zeta}(t) = F_j \, \zeta(t) + v_{ji}(t) & \text{if } \tilde{q} \neq \{q_i\} \end{cases} \qquad (17)$$

where $F_j = A_j - G_j C_j$ and $v_{ji}(t) = [(A_j - A_i) - G_j (C_j - C_i)] \, x(t) + (B_j - B_i) u(t)$.

2. to each discrete transition between locations in the subset $Q \times E_{\mathcal{O}}$ the following resets are applied:

(a) for transitions $(q_j, \{q_\ell\}) \to (q_i, \{q_\ell\})$, with $q_i \neq q_j$, occuring at times $t_k \neq \hat{t}_k$, the continuous state (x, ζ) is subject to the reset

$$x(t_k) = R_{ji}^1 \, x(t_k^-) + R_{ji}^0 \qquad (18)$$
$$\zeta(t_k) = \zeta(t_k^-) - R_{ji}^0 + [I - R_{ji}^1] x(t_k^-) \quad (19)$$

(b) for transitions $(q_\ell, \{q_j\}) \to (q_\ell, \{q_i\})$, with $\{q_i\} \neq \{q_j\}$, occuring at times $\hat{t}_k \neq t_k$, only the component ζ of the continuous state is reset according to

$$\zeta(\hat{t}_k) = R_{ji}^1 \zeta(\hat{t}_k^-) + R_{ji}^0 - [I - R_{ji}^1] x(\hat{t}_k) \; (20)$$

(c) for transitions $(q_j, \{q_j\}) \to (q_i, \{q_i\})$, with $q_i \neq q_j$, occuring at times $t_k = \hat{t}_k$, the continuous state (x, ζ) is subject to the resets (18) and

$$\zeta(t_k) = R_{ji}^1 \zeta(t_k^-) \qquad (21)$$

Notice that the description above is complete in the sense that if the hybrid plant $\mathcal{H}_P$ and the hybrid observer $\mathcal{H}_O$ make a transition synchronously at some time $t_k = \hat{t}_k$, then necessarily the transition is between plant locations correctly identified by the hybrid observer. This is because of the dwell-time hypothesis and the assumption of identification within time $\Delta < D$.

Dynamics (16–17) are readily obtained from (4) and (14). While resets (19) and (20) are given by (6) and (15).

Let us introduce the following notation (see [11]): $\|M\|$ and $\|M\|_\infty$ are the l_2 and the l_∞ norm of a matrix (or vector) M, respectively. $\|m(t)\|_\infty = \max_{k=1,q} \sup_{t \geq 0} |m_k(t)|$, the L_∞ norm of q–dimensional signals $m : \mathbb{R} \to \mathbb{R}^q$; and $\|m(t)\|_1 = \max_{i=1,q} \sum_{j=1,q} \int_0^\infty |m_{ij}(\tau)| \, d\tau$, the L_1 norm of $q \times q$–dimensional signals $m : \mathbb{R} \to \mathbb{R}^{q \times q}$. Moreover, given a square matrix A, let $\alpha(A) = \max\{\mathrm{Re}(\lambda) \,|\, \lambda \text{ such that } \det(A - \lambda I) = 0\}$ denote the spectral abscissa of A and $k(A) = \|T\| \, \|T^{-1}\|$, with T such that $T^{-1} A T$ is in the Jordan canonical form.

[3] The Luenberger observers (10) contained in the residual generators, which are designed to converge to the same state variable x, do not provide a satisfactory estimate of the evolution of x since they are tuning according to (13) in order to meet the specification of producing a residual with a transient time less than Δ. Hence, they exhibit a large overshoot which is undesirable for feedback purpose.

Theorem 6 *Given a hybrid system $\mathcal{H}_P$ as in (1–6) with dwell time D such that matrices A_i in (4) have dinstict eigenvalues for each i such that $\{q_i\} \in E_{\mathcal{O}}$, if for each $\{q_i\} \in E_{\mathcal{O}}$ there exist gain matrices G_i such that*

1. *$F_i = A_i - G_i C_i$ has distint eigenvalues;*
2. *the location observer identifies a change in the hybrid system location within time Δ with $0 \leq \Delta < D$;*
3. $\alpha(F_i) + \dfrac{\max\{0, \log[r_i^1\, k(F_i)]}{D - \Delta} \leq -\mu < 0$

where $r_i^1 = \max_{q_j \in Reach(q_i)} \|R_{ij}^1\|$, then the hybrid observer $\mathcal{H}_O$ is exponentially ultimately bounded with rate of convergence μ.

By choosing Δ small enough, the ultimate bound can be made as close as desired to the threshoold value $\max_{\{q_j\} \in E_{\mathcal{O}}} \max_{q_i \in Reach(q_j)} \left\{k(F_j)\, \left(\|R_{ji}^0\| + \|I - R_{ji}^1\| \cdot \|x(t)\|_\infty\right)\right\}$. In the case of absence of continuous state resets, any desired value for the ultimate bound can be achieved for Δ small enough.

The following results will be used in the proof of Theorem 6.

Lemma 7 ([7]) *Let $A \in \mathbb{R}^{n \times n}$ be a matrix with distinct eigenvalues. Then*

$$\|e^{At}\| \leq k(A)\, e^{\alpha(A) t} \qquad \forall t \geq 0 \qquad (22)$$

If $F \in \mathbb{R}^{n \times n}$ is another matrix with distinct eigenvalues and $\alpha(F) < \alpha(A) < 0$, then, $\forall t \geq 0$,

$$\|I - e^{At}\| \leq k(A) \|A\| t \qquad (23)$$
$$\|e^{Ft} - e^{At}\| \leq k(F) k(A) (\|F\| + \|A\|) t \qquad (24)$$

Proof of Theorem 6. Let $B_x > 0$ and $B_u > 0$, such that $\|x(t)\|_\infty \leq B_x$ and $\|u(t)\|_\infty \leq B_u$, so that

$$\|v_{ji}(t)\|_\infty \leq V_{ji} \quad \forall\, \{q_j\} \in E_{\mathcal{O}}, q_i \in \text{Reach}(q_j)$$

with $V_{ji} = \|\,[(A_j - A_i) - G_j(C_j - C_i)]\,\|_\infty B_x + \|B_j - B_i\|_\infty B_u$.

Consider two subsequent transitions of the hybrid plant $\mathcal{H}_P$, occurring at times t_{k-1} and t_k respectively.

Let $q = q_j$ for $t \in [t_{k-1}, t_k)$ and $q = q_i \in \text{Reach}(q_j)$ for $t \in [t_k, t_{k+1})$. By the hypothesis on dwell time equal to D, $t_k - t_{k-1} \geq D$. Since by hypothesis 2, $\Delta \leq D - \beta$, then the location observer identifies the $k-1$-th and $k+1$-th state transitions at some times $\hat{t}_{k-1}$ and $\hat{t}_k$, respectively, with $0 \leq \hat{t}_{k-1} - t_{k-1} \leq \Delta$ and $0 \leq \hat{t}_k - t_k \leq \Delta$. That is: $\tilde{q} = \{q_j\}$ for $t \in [\hat{t}_{k-1}, \hat{t}_k)$ and $\tilde{q} = \{q_i\}$ for $t \in [\hat{t}_k, \hat{t}_{k+1})$. Furthermore, notice that, by the hypothesis on the dwell time and hypothesis 2, $\hat{t}_k - \hat{t}_{k-1} \geq t_k - \hat{t}_{k-1} \geq D - \Delta \geq \beta > 0$.

Consider the time interval $[\hat{t}_{k-1}, \hat{t}_k)$. The composed hybrid system $\mathcal{H}_P \otimes \mathcal{H}_O$ lies on

- location $(q_j, \{q_j\})$ for $t \in [\hat{t}_{k-1}, t_k)$, and
- location $(q_i, \{q_j\})$ for $t \in [t_k, \hat{t}_k)$.

Since in the time interval $[\hat{t}_{k-1}, t_k)$ the location observer properly identifies the plant location q_j, then according to (17) the ζ is subject to an autonomous dynamic that convergences to zero if the matrix F_j is Hurwitz for some choice of the observer gain G_j. In particular, we assume that observer gains G_j satisfying hyphotesis 3, for some $\beta \in (0, D)$, can be selected.

However, due to the mismatch between the plant location and the observer location for $t \in [t_k, \hat{t}_k)$, $\zeta(t)$ may fail to converge later due to the injection of the disturbance $v_{ji}(t)$ in (17). Hence, the convergent behavior for $t \in [\hat{t}_{k-1}, t_k)$ has to compensate the divergent behavior for $t \in [t_k, \hat{t}_k)$.

By integrating (17), we have

$$\zeta(t) = e^{F_j (t - \hat{t}_{k-1})} \zeta(\hat{t}_{k-1}) \;\; \forall\, t \in [\hat{t}_{k-1}, t_k) \qquad (25)$$

and, by (19),

$$\zeta(t_k) = \zeta(t_k^-) - R_{ji}^0 + [I - R_{ji}^1] x(t_k^-)$$
$$= e^{F_j (t_k - \hat{t}_{k-1})} \zeta(\hat{t}_{k-1}) - R_{ji}^0 + [I - R_{ji}^1] x(t_k^-) \quad (26)$$

Hence, by integrating (17), for $t \in [t_k, \hat{t}_k)$ we have

$$\zeta(t) \;\; = \;\; e^{F_j (t - \hat{t}_{k-1})} \zeta(\hat{t}_{k-1}) - e^{F_j (t - t_k)} R_{ji}^0$$

$$+e^{F_j(t-t_k)}[I-R^1_{ji}]x(t_k^-)$$
$$+\int_0^{t-t_k} e^{F_j(t-t_k-\tau)} v_{ji}(\tau+t_k)\,d\tau \quad (27)$$

$\forall\, t \in [t_k, \hat{t}_k)$. Finally, by (20)

$$\zeta(\hat{t}_k) = R^1_{ji}\,\zeta(\hat{t}_k^-) + R^0_{ji} - [I-R^1_{ji}]\,e^{A_i(\hat{t}_k-t_k)}x(t_k)$$
$$-[I-R^1_{ji}]\int_0^{\hat{t}_k-t_k} e^{A_i(\hat{t}_k-t_k-\tau)}B_i u(\tau+t_k)\,d\tau$$
$$= R^1_{ji}\,e^{F_j(\hat{t}_k-\hat{t}_{k-1})}\zeta(\hat{t}_{k-1}) + [\,(I-e^{A_i(\hat{t}_k-t_k)})$$
$$-R^1_{ji}\,(e^{F_j(\hat{t}_k-t_k)}-e^{A_i(\hat{t}_k-t_k)})\,]R^0_{ji} + \{\,[I-R^1_{ji}]$$
$$(e^{F_j(\hat{t}_k-t_k)}-e^{A_i(\hat{t}_k-t_k)})R^1_{ji} + [(I-e^{F_j(\hat{t}_k-t_k)})R^1_{ji}$$
$$-R^1_{ji}\,(I-e^{F_j(\hat{t}_k-t_k)})]\,\}x(t_k^-) - [I-R^1_{ji}]$$
$$\int_0^{\hat{t}_k-t_k} e^{A_i(\hat{t}_k-t_k-\tau)}B_i u(\tau+t_k)\,d\tau$$
$$+R^1_{ji}\int_0^{\hat{t}_k-t_k} e^{F_j(\hat{t}_k-t_k-\tau)}v_{ji}(\tau+t_k)\,d\tau \quad (28)$$

Then, an upper bound for the norm of the observation error $\|\zeta\|$ at each time $\hat{t}_k$ is obtained as follows.

Since by hypothesis 1 the observation matrices $F_j = A_j - G_jC_j$ have distint eigenvalues, then by (22) we have $\|R^1_{ji}\,e^{F_j(\hat{t}_k-\hat{t}_{k-1})}\zeta(\hat{t}_{k-1})\| \le r^1_j\,k(F_j)\,e^{\alpha(F_j)(\hat{t}_k-\hat{t}_{k-1})}\|\zeta(\hat{t}_{k-1})\|$. Moreover, since $\hat{t}_k-\hat{t}_{k-1}\ge\beta$, then by condition 3 it follows that

$$r^1_j\,k(F_j)\,e^{\alpha(F_j)(\hat{t}_k-\hat{t}_{k-1})} = e^{\log[r^1_j\,k(F_j)]+\alpha(F_j)(\hat{t}_k-\hat{t}_{k-1})}$$
$$\le e^{\left[\alpha(F_j)+\frac{\max\{0,\log[r^1_j\,k(F_j)]\}}{\beta}\right](\hat{t}_k-\hat{t}_{k-1})} \le e^{-\mu(\hat{t}_k-\hat{t}_{k-1})}$$

$$\|R^1_{ji}\,e^{F_j(\hat{t}_k-\hat{t}_{k-1})}\zeta(\hat{t}_{k-1})\| \le e^{-\mu(\hat{t}_k-\hat{t}_{k-1})}\|\zeta(\hat{t}_{k-1})\| \quad (29)$$

By (23–24), terms right multiplied by R^0_{ji} and $x(t_k^-)$ are upper bounded by

$$[\,k(A_i)\,\|A_i\| + k(A_i)k(F_j)(\|A_i\|+\|F_j\|)$$
$$\|R^1_{ji}\|\,]\,\|R^0_{ji}\|\,\Delta = b^1_{ji}\Delta \quad (30)$$
$$[\,k(A_i)\,k(F_j)(\|A_i\|+\|F_j\|)\|I-R^1_{ji}\|\,\|R^1_{ji}\|$$
$$+2k(F_j)\|F_j\|\|R^1_{ji}\|\,]\,B_x\Delta = b^2_{ji}B_x\Delta \quad (31)$$

respectively. Furthermore, for the v_{ji} forced term, since by hypothesis 2, $\hat{t}_k - t_k \le \Delta$, then

$$\left\|R^1_{ji}\int_0^{\hat{t}_k-t_k} e^{F_j(\hat{t}_k-t_k-\tau)}v_{ji}(\tau+t_k)\,d\tau\right\|$$
$$\le \|R^1_{ji}\|k(F_j)\sup_{t\ge0}\|v_{ji}(t)\|\int_0^{\hat{t}_k-t_k} e^{\alpha(F_j)\tau}\,d\tau$$
$$\le \|R^1_{ji}\|\sqrt{n}k(F_j)V_{ji}\Delta = b^3_{ji}V_{ji}\Delta \quad (32)$$

Finally, by the same arguments, the $u(t)$ forced term is bounded by

$$\|I-R^1_{ji}\|\sqrt{n}k(A_i)\|B_i\|B_u\Delta = b^4_{ji}B_u\Delta \quad (33)$$

By using (29–33), the norm of the observation error is upper bounded at each observer transition time $\hat{t}_k$ as follows:

$$\|\zeta(\hat{t}_k)\| \le e^{-\mu(\hat{t}_k-\hat{t}_{k-1})}\|\zeta(\hat{t}_{k-1})\| + b\Delta \quad (34)$$

$$b = \max_{\substack{\{q_j\}\in E_{\mathcal{O}}\\ q_i\in \mathrm{Reach}(q_j)}} \left\{b^1_{ji} + b^2_{ji}B_x + b^3_{ji}V_{ji} + b^4_{ji}B_u\right\}$$

Since $\hat{t}_k-\hat{t}_{k-1}\ge\beta$ and $e^{-\mu\beta}<1$, then from (34) we obtain

$$\|\zeta(\hat{t}_k)\| \le e^{-\mu(\hat{t}_k-\hat{t}_K)}\left\|\zeta(\hat{t}_K)\right\| + b\Delta\sum_{h=K}^{k} e^{-\mu(\hat{t}_k-\hat{t}_h)}$$
$$\le e^{-\mu(\hat{t}_k-\hat{t}_K)}\left\|\zeta(\hat{t}_K)\right\| + b\Delta\sum_{h=K}^{k} e^{-\mu(k-h)\beta}$$
$$= e^{-\mu(\hat{t}_k-\hat{t}_K)}\left\|\zeta(\hat{t}_K)\right\| + \frac{b\Delta}{1-e^{-\mu\beta}} \quad (35)$$

The above inequality shows that the value of the norm of the observation error after each location observer transition is upper bounded by an exponential with rate $-\mu$ that converges to the value $\frac{b\Delta}{1-e^{-\mu D}}$.

Substituting the upper bound (35) into (25), we have

$$\|\zeta(t)\| \le k(F_j)\,e^{-\mu(t-\hat{t}_K)}\|\zeta(\hat{t}_K)\| + \frac{k(F_j)b\Delta}{1-e^{-\mu\beta}} \quad (36)$$

$\forall\ t\in[\hat{t}_{k-1},t_k)$. Moreover, substituting (35) into (27), we have

$$\|\zeta(t)\| \le k(F_j)e^{-\mu(t-\hat{t}_K)}\|\zeta(\hat{t}_K)\| + k(F_j)\left\{\frac{b}{1-e^{-\mu\beta}}\right.$$
$$\left.+\sqrt{n}V_{ji}\right\}\Delta + k(F_j)\left(\|R^0_{ji}\| + \|I-R^1_{ji}\|B_x\right) \quad (37)$$

$\forall\ t\in[t_k,\hat{t}_k)$. An upper bound for the evolution of $\zeta(t)$, for $t\ge\hat{t}_K$ and any switching sequence of plant locations, is now obtained comparing the expressions (36) and (37):

$$\|\zeta(t)\| \leq c_2 e^{-\mu(t-t_K)} \|\zeta(t_K)\| + c_1\Delta + c_0$$

where $c_2 = \max_{\{q_j\}\in E_{\mathcal{O}}}\{k(F_j)\}$ and

$$c_1 = \max_{\substack{\{q_j\} \in E_{\mathcal{O}} \\ q_i \in \mathrm{Reach}(q_j)}} \left\{ k(F_j) \left(\frac{b}{1-e^{-\mu\beta}} + \sqrt{n} V_{ji} \right) \right\}$$

$$c_0 = \max_{\substack{\{q_j\} \in E_{\mathcal{O}} \\ q_i \in \mathrm{Reach}(q_j)}} \left\{ k(F_j) \left(\|R^0_{ji}\| + \|I - R^1_{ji}\| B_x \right) \right\}$$

This proves exponential ultimate boundness of the hybrid observer to the ultimate bound $c_1\Delta + c_0$. By choosing Δ small enough, the minimum lower bound c_0 can be approached.

Finally, in the case of absence of continuous state resets, i.e. $R^0_{ji} = 0$ and $R^1_{ji} = I$ for every $\{q_j\} \in E_{\mathcal{O}}$ and $q_i \in \mathrm{Reach}(q_j)$, any given ultimate bound can be attained by choosing an appropriate Δ. ■

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