Homework 4 Solution

EE 290n - Advanced Topics in Systems Theory

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1. For two posets A and B, a function $f : A \to B$ is an **order embedding** if $a \le a' \iff f(a) \le f(a')$. A function $f : A \to B$ is an **order isomorphism** if it is onto and an order embedding.

   Recall that a function $f : A \to B$ is **one-to-one** (or **injective**) if for all $a, a' \in A$,

   $$a \ne a' \Rightarrow f(a) \ne f(a').$$

   That is, no two distinct values in the domain yield distinct values in the codomain.

   Given a function $f : A \to B$ let $\hat{f} : \wp(A) \to \wp(B)$ denote the **image function**, defined by

   $$\forall A' \subset A, \quad \hat{f}(A') = \{b \in B \mid \exists a \in A' \text{ such that } f(a) = b\}.$$

   In words, $\hat{f}$ takes a set of arguments to f and returns the set of results. The returned result $\hat{f}(A')$ is called the **image** of A' under function f. The **range** of a function $f : A \to B$ is simply the image of its domain, $\hat{f}(A)$.

   A function $f : A \to B$ is **onto** (or **surjective**) if $\hat{f}(A) = B$.

   (a) Show that if $f : A \to B$ is an order embedding, then f is one-to-one.

   (b) Show that if $f : A \to B$ is an order isomorphism, then there is an order isomorphism $g : B \to A$.

   Solution.

   (a) We need to show that
   $$a \ne a' \Rightarrow f(a) \ne f(a').$$

   Equivalently, we can show that

   $$f(a) = f(a') \Rightarrow a = a'.$$

   Note that
   $$f(a) = f(a') \Rightarrow f(a) \le f(a')$$

   and

   $$f(a) = f(a') \Rightarrow f(a') \le f(a).$$

   Since f is an order embedding, this implies that $a \le a'$ and $a' \le a$, which together imply that $a = a'$.

   (b) Using the result from part (a), f is one-to-one and onto, so f^{-1} is a well-defined function and is also one-to-one and onto. Let $g = f^{-1}$. We can show that g is an order embedding, and hence an order isomorphism. To show that, note that since f is an order embedding,

   $$f(a) \le f(a') \iff a \le a',$$

which, letting $b = f(a)$ and $b' = f(a')$, implies that

$$b \leq b' \iff g(b) \leq g(b').$$

Thus, g is an order embedding.

□

2. Consider the model shown in figure 1. This model is accessible at the following URL:

`http://embedded.eecs.berkeley.edu/concurrency/lectures/Models11/zeno.xml`

(a) Prove that this system is not discrete.

(b) Construct a model in the DE domain in Ptolemy II similar to the one in figure 1 with the following properties:

    i. it has a feedback loop where no actor is delta causal,

    ii. time diverges (it has events at times greater than any finite time), and

    iii. the system is discrete (there is no Zeno condition).

This demonstrates that the condition requiring a delta-causal actor in a feedback loop is only sufficient, not necessary, to prevent Zeno conditions.

Solution.

(a) The times at which output events (going to the TimedPlotter) are produced by this model are *Naturals* $\cup A$, where

$$A = \{\sum_{i=1}^{n} 1/i^2 \mid n \in \textit{Naturals}\},$$

where the times in *Naturals* come from the Clock actor and the ones in A are generated by the feedback loop. It can be shown that for all $n \in \textit{Naturals}$,

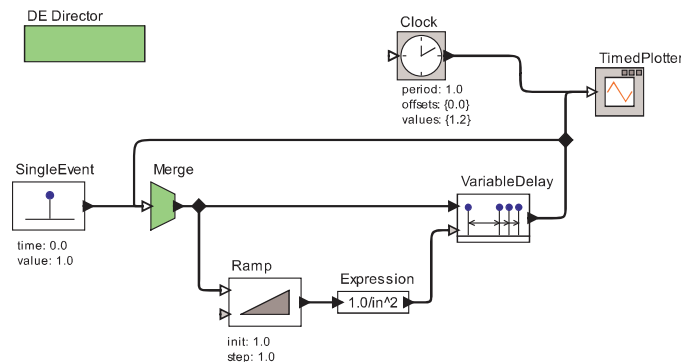$$\sum_{i=1}^{n} 1/i^2 < \sum_{i=1}^{\infty} 1/i^2 = \pi^2/6 \approx 1.6449.$$

Figure 1: A discrete-event model that exhibits Zeno behavior.

Hence, there are an infinite number of events prior to time 2.0, where there is an event from the clock. We can now show that there can be no order embedding to the integers. Suppose to the contrary that f is such an order embedding. Let $f(2.0) = m$ be a finite integer and $f(1.0) = n$ be another finite integer. Since f is an order embedding, it must be one-to-one (see problem 1), hence $n \neq m$. Moreover, since $1.0 < 2.0$, it must be that $f(1.0) < f(2.0)$. Since f is an order embedding (and hence is one-to-one), there can be at most $m - n - 1$ events (a finite number) between times 1.0 and 2.0. However, all events in A except the first are in this range, and the number of such events is infinite. So f cannot be an order embedding.

(b) A model is shown in figure 2. Here, the events generated by the feedback loop are in

$$B = \{\sum_{i=1}^{n} 1/i \mid n \in \textit{Naturals}\},$$

There are now a finite number of such events between any two events of the Clock actor, so it is easy to construct an order embedding.

$\square$

3. Given a metric space (A, d), show that for all $a, b \in A$

$$d(a, b) \geq 0.$$

Solution. Letting $c = a$ in the triangle inequality (property 3), we get

$$d(a, b) + b(b, a) \geq d(a, a).$$

Property 2 now implies that
$$d(a, b) + b(b, a) \geq 0.$$

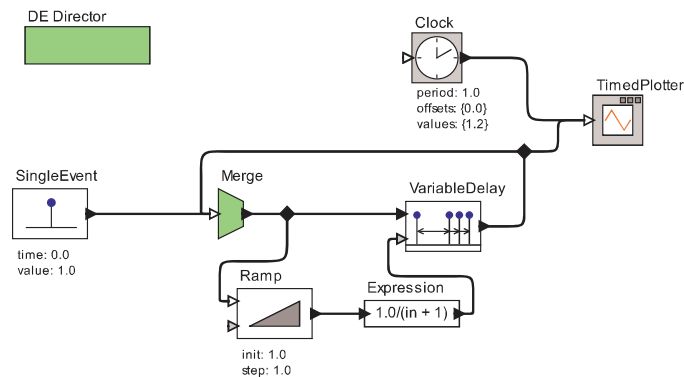Since $d(a, b) = d(b, a)$ (property 1), this leads to

$$2d(a, b) \geq 0$$

Figure 2: A discrete-event model similar to that in figure 1 that does not exhibit Zeno behavior.

or
$$d(a,b) \geq 0,$$

the desired result. □

4. Prove that an ultrametric is a metric.

   Solution. The first two properties of an ultrametric are identical to the first two properties of a metric. So we only need to show that the third property of an ultrametric implies the third property of a metric. Using methods similar to problem 4, we can show that for an ultrametric, $d(a,b) \geq 0$. Then from the ultrametric property

   $$\max(d(a,b), d(b,c)) \geq d(a,c),$$

   since all the distances are non-negative, it is clearly true that

   $$d(a,b) + d(b,c) \geq d(a,c).$$

   □