# Homework 3

EE 290n - Advanced Topics in Systems Theory

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1. Let $(A, \leq)$ and $(B, \leq)$ be CPOs. We can form a poset $(A \times B, \leq)$ where the order is a **lexicographic order**, where for all $(a_1, b_1), (a_2, b_2) \in A \times B$,

$$(a_1, b_1) \leq (a_2, b_2) \iff (a_1 = a_2 \text{ and } b_1 \leq b_2) \text{ or } (a_1 \neq a_2 \text{ and } a_1 \leq a_2).$$

   With this order, determine whether $(A \times B, \leq)$ is a CPO. Prove that it is or is not.

2. In the definition of a **sequential** function, we need to first assume that the function is continuous. This problem explores why that assumption is necessary. In particular, let $A = T^\omega$ for some set $T$ and assume that $f : A^n \to A^m$ is a function for some natural numbers $n$ and $m$ where for all $a, b \in A^n$ such that $a \sqsubseteq b$, there exists an $i \in \{1, \ldots, n\}$ such that

$$\pi_i(a) = \pi_i(b) \Rightarrow f(a) = f(b).$$

   Show that, by itself, this is not enough to guarantee that $f$ is continuous for any $n$.

3. The LUB of a chain can be interpreted as a limit, but it is not always the limit that one would expect intuitively. This problem explores this issue for the set $T = \{t, f\}$ and the CPO $(A = T^\omega, \sqsubseteq)$, and offers an alternative way to construct limits.

   (a) As a warm up, find the LUB of the following set, or show that it has no LUB:

$$\{(t), (t, f), (t, f, t), (t, f, t, f), \ldots\}.$$

   (b) Show that the following set has no LUB:

$$\{a_1, a_2, a_3, \ldots\} = \{(t, f), (t, t, f), (t, t, t, f), \ldots\}.$$

   (c) Given a set $A$, a metric is a function $d \colon A \times A \to \mathit{Reals}$ such that three properties are satisfied for all $a, b, c \in A$:

      i. $d(a, b) = d(b, a)$,
      ii. $d(a, a) = 0$ and
      iii. $d(a, b) + d(b, c) \geq d(a, c)$.

   With $A = T^\omega$, we can define a metric $d$ where for all $a, b \in A$,

$$d(a, b) = 1/n$$

   where $n$ is the index of the first position at which $a$ and $b$ differ, and is zero if $a = b$. For example, $d((t), (f)) = 1$ and $d((t), (t, f)) = 1/2$. Show that $d$ is a metric.

(d) For the set in part (b), $a_i$ is defined to be the sequence starting with $i$ instances of $t$ followed by one $f$. Let

$$a = (t,t,t,...)$$

(an infinite sequence of $t$). Show that $a_i$ converges to $a$ as $i \to \infty$ in the sense that for any real number $\varepsilon > 0$, there is an integer $K$ such that for all $i > K$,

$$d(a_i, a) < \varepsilon.$$

4. In process networks, a two-input, two-output identity function is not implementable as a sequential function. However, using a well-designed nondeterministic merge, a deterministic two-input, two-output identity function can be defined as a composite actor. Use or adapt the nondeterministic merge that you created in the previous homework to build such a composite actor, and construct models that demonstrate that it properly functions as an identity function.