Homework 3

EE 290n - Advanced Topics in Systems Theory

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1. Let $(A, \leq)$ and $(B, \leq)$ be CPOs. We can form a poset $(A \times B, \leq)$ where the order is a **lexicographic order**, where for all $(a_1, b_1), (a_2, b_2) \in A \times B$,

$$(a_1, b_1) \leq (a_2, b_2) \iff a_1 \leq a_2 \text{ or } (a_1 = a_2 \text{ and } b_1 \leq b_2).$$

With this order, determine whether $(A \times B, \leq)$ is a CPO. Prove that it is or is not.

Solution. It is a CPO. It is straightforward to show that it is a poset, and that it has a least element, $(\perp_A, \perp_B)$, assuming A and B are posets with least elements. We need to show that every chain C has a LUB. Consider a chain $C = \{(a_n, b_n) \mid n \in Naturals\}$. Note that $\hat{\pi}_1(C) = \{a_n \mid n \in Naturals\}$, is a chain in A, by the definition of the lexicographic order. Hence it has a LUB, which we denote $a = \bigvee \hat{\pi}_1(C)$. If $a \notin \hat{\pi}_1(C)$, then

$$\bigvee C = (a, \perp_B).$$

If $a \in \hat{\pi}_1(C)$, then define the set

$$Q = \{b \in \hat{\pi}_2(C) \mid (a, b) \in C\}.$$

By the definition of the lexicographic order, this is a chain in B, and hence has a LUB. In this case,

$$\bigvee C = (a, \bigvee Q).$$

□

2. In the definition of a **sequential** function, we need to first assume that the function is continuous. This problem explores why that assumption is necessary. In particular, let $A = T^{**}$ for some set T and assume that $f : A^n \to A^m$ is a function for some natural numbers n and m where for all $a, b \in A$ such that $a \sqsubseteq b$, there exists an $i \in \{1, \ldots, n\}$ such that

$$\pi_i(a) = \pi_i(b) \Rightarrow f(a) = f(b).$$

Show that, by itself, this is not enough to guarantee that f is continuous for any n.

Solution. Consider first $n = 1$. In this case, every function satisfies the given constraint, including functions that are not continuous. So in this case, the specified condition does not guarantee that the function is continuous.

Suppose we have a non-continuous function $g \colon A \to A^m$. Then for any natural number n we can define a function $f \colon A^n \to A^m$ by

$$\forall\, a \in A^n, \quad f(a) = g(\pi_1(a)).$$

This function always satisfies the specified condition (choosing $i = 1$), and is clearly not continuous. □

3. The LUB of a chain can be interpreted as a limit, but it is not always the limit that one would expect intuitively. This problem explores this issue for the set $T = \{t, f\}$ and the CPO $(A = T^{**}, \sqsubseteq)$, and offers an alternative way to construct limits.

   (a) As a warm up, find the LUB of the following set, or show that it has no LUB:

   $$\{(t), (t, f), (t, f, t), (t, f, t, f), ...\}.$$

   (b) Show that the following set has no LUB:

   $$\{a_1, a_2, a_3, ...\} = \{(t, f), (t, t, f), (t, t, t, f), ...\}.$$

   (c) Given a set A, a metric is a function $d\colon A \times A \to \mathit{Reals}$ such that four properties are satisfied for all $a, b, c \in A$:

      i. $d(a, b) = d(b, a)$,
      ii. $d(a, a) = 0$
      iii. $d(a, b) \geq 0$, and
      iv. $d(a, b) + d(b, c) \geq d(a, c)$.

   With $A = T^{**}$, we can define a metric d where for all $a, b \in A$,

   $$d(a, b) = 1/n$$

   where n is the index of the first position at which a and b differ, and is zero if $a = b$. For example, $d((t), (f)) = 1$ and $d((t), (t, f)) = 1/2$. Show that d is a metric.

   (d) For the set in part (b), a_i is defined to be the sequence starting with i instances of t followed by one f. Let

   $$a = (t, t, t, ...)$$

   (an infinite sequence of t). Show that a_i converges to a as $i \to \infty$ in the sense that for any real number $\varepsilon > 0$, there is an integer K such that for all $i > K$,

   $$d(a_i, a) < \varepsilon.$$

Solution.

   (a) The LUB is simply the infinite sequence of alternating t and f,

   $$(t, f, t, f, t, f, ...).$$

   (b) Suppose the set has an upper bound. Then every element of the set must be a prefix of that upper bound. Hence, the first two elements of the set must be t and f, since (t, f) is a prefix of the upper bound. But so is (t, t, f), so the first two elements must be t, a contradiction.

   (c) The first three properties of a metric are trivially satisfied by d. The only non-trivial one is the last, the triangle inequality. Suppose that $d(a, b) = 1/n$ and $d(b, c) = 1/m$. Then

a and b differ first at position n and b and c differ first at position m. Clearly, a and c must be identical up to position $\min(n,m)$, hence

$$d(a,c) \leq 1/\min(n,m).$$

We only then have to show that

$$1/n + 1/m \geq 1/\min(n,m).$$

This is true because $1/n \geq 0$, $1/m \geq 0$, and each of

$$1/m \geq 1/\min(n,m)$$

and

$$1/n \geq 1/\min(n,m).$$

(d) The distance between a_i and a is

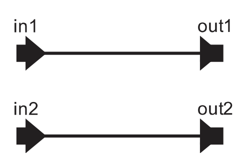
$$d(a_i,a) = 1/(i+1),$$

which can be made as small as we like by choosing i large enough.

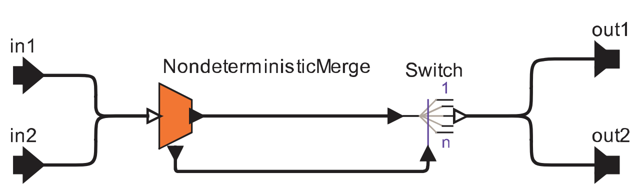
□

4. In process networks, a two-input, two-output identity function is not implementable as a sequential function. However, using a well-designed nondeterministic merge, a deterministic two-input, two-output identity function can be defined as a composite actor. Use or adapt the nondeterministic merge that you created in the previous homework to build such a composite actor, and construct models that demonstrate that it properly functions as an identity function.

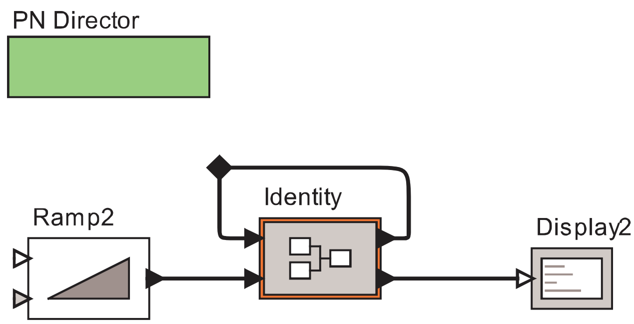
Solution. A trivial composite actor that performs the two-input, two-output identity function is shown below:



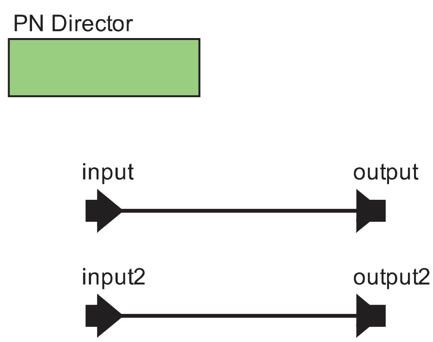
A more interesting solution that gets to the spirit of the problem is:



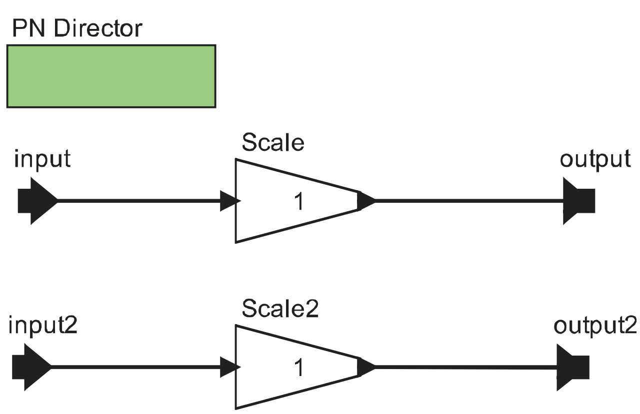
That the behavior is correct is tested with the following model:



Note that I tried some other solutions as well, but they failed due to limitations of the implementation of PN in Ptolemy II (as of version 4.0.1). The following solution attempts to make the trivial solution above less trivial by making the composite actor opaque (by giving it its own director):



This solution yields an obscure exception, indicating that the code was written without anticipating such models. The following variant gets around this problem, but still fails to work properly. Only one of the two input streams makes it to the output.



□