



# Concurrent Models of Computation for Embedded Software

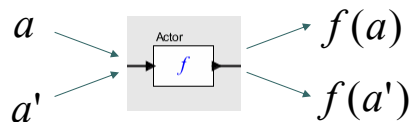
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Lecture 6: Process Networks Semantics

PN Semantics  
Where This is Going



A signal is a sequence of values  
Define a prefix order:

$$a \sqsubseteq a'$$

means that  $x$  is a prefix of  $y$ .

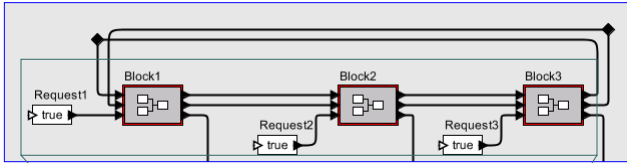
Actors are *monotonic* functions:

$$a \sqsubseteq a' \Rightarrow f(a) \sqsubseteq f(a')$$

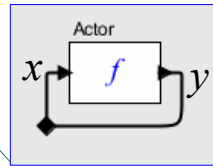
Stronger condition: Actors are *continuous* functions  
(intuitively: they don't wait forever to produce outputs).

## PN Semantics of Composition (Kahn, '74) This Approach to Semantics is “Tarskian”

If the components are deterministic, the composition is deterministic.



$$x = y \Rightarrow$$
$$f(x) = x$$



Fixed point theorem:

- Continuous function has a unique least fixed point
- Execution procedure for finding that fixed point
- Successive approximations to the fixed point

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## What is Order?

Intuition:

1.  $0 < 1$
2.  $1 < \infty$
3. child  $<$  parent
4. child  $>$  parent
5. 11,000/3,501 is a better approximation to  $\pi$  than 22/7
6. integer  $n$  is a divisor of integer  $m$ .
7. Set  $A$  is a subset of set  $B$ .

Which of these are *partial orders*?

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## Relations

- A *relation*  $R$  from  $A$  to  $B$  is a subset of  $A \times B$
- A *function*  $F$  from  $A$  to  $B$  is a relation where
$$(a, b) \in R \text{ and } (a, b') \in R \Rightarrow b = b'$$
- A *binary relation*  $R$  on  $A$  is a subset of  $A \times A$
- A *binary relation*  $R$  on  $A$  is *reflexive* if
$$\forall a \in A, (a, a) \in R$$
- A *binary relation*  $R$  on  $A$  is *symmetric* if
$$(a, b) \in R \Rightarrow (b, a) \in R$$
- A *binary relation*  $R$  on  $A$  is *antisymmetric* if
$$(a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b$$
- A *binary relation*  $R$  on  $A$  is *transitive* if
$$(a, b) \in R \text{ and } (b, c) \in R \Rightarrow (a, c) \in R$$

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## Infix Notation for Binary Relations

- $(a, b) \in R$  can be written  $a R b$
- A symbol can be used instead of  $R$ . For examples:
  - $\leq \subset N \times N$  is a relation.
  - $(a, b) \in \leq$  is written  $a \leq b$
- A function  $f \in (A, B)$  can be written  $f: A \rightarrow B$

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## Partial Orders

A *partial order* on the set  $A$  is a binary relation  $\leq$  that is:

For all  $a, b, c \in A$ ,

- reflexive:  $a \leq a$
- antisymmetric:  $a \leq b$  and  $b \leq a \Rightarrow a = b$
- transitive:  $a \leq b$  and  $b \leq c \Rightarrow a \leq c$

A *partially ordered set (poset)* is a set  $A$  and a binary relation  $\leq$ , written  $(A, \leq)$ .

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## Strict Partial Order

For every partial order  $\leq$  there is a *strict partial order*  $<$  where  $a < b$  if and only if  $a \leq b$  and  $a \neq b$ .

A *strict poset* is a set and a strict partial order.

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## Total Orders

Elements  $a$  and  $b$  of a poset  $(A, \leq)$  are *comparable* if either  $a \leq b$  or  $b \leq a$ . Otherwise they are *incomparable*.

A poset  $(A, \leq)$  is *totally ordered* if every pair of elements is comparable.

Totally ordered sets are also called *linearly ordered sets* and *chains*.

A *well-ordered set* is a chain such that every non-empty subset has a least element.

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## Quiz

1. Is the set of integers with the usual numerical ordering a well-ordered set?
2. Given a set  $A$  and its *powerset* (set of all subsets)  $P(A)$ , is  $(P(A), \subseteq)$  a poset? A chain?
3. For  $A = \{a, b, c\}$  (a set of three letters), find a well-ordered subset of  $(P(A), \subseteq)$ .

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## Answers

1. Is the set of integers with the usual numerical ordering a well-ordered set?  
*No. The set itself is a chain with no least element.*
2. Given a set  $A$  and its powerset (set of all subsets)  $P(A)$ , is  $(P(A), \subseteq)$  a poset? A chain?  
*It is a poset, but not a chain.*
3. For  $A = \{a, b, c\}$  (a set of three letters), find a well-ordered subset of  $(P(A), \subseteq)$ .  
*One possibility:  $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$*

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## Pertinent Example: Prefix Orders

Let  $A$  be a type (a set of values).

Let  $A^{**}$  be the set of all finite and infinite sequences of elements of  $A$ , including the empty sequence  $\perp$  (bottom).

Let  $\sqsubseteq$  be a binary relation on  $A^{**}$  such that  $a \sqsubseteq b$  if  $a$  is a *prefix* of  $b$ . That is, for all  $n$  in  $N$  such that  $a(n)$  is defined, then  $b(n)$  is defined and  $a(n) = b(n)$ .

This is called a *prefix order*.

During execution, any output of a PN actor is a well-ordered subset of  $(A^{**}, \sqsubseteq)$ .

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## Join (Least Upper Bound)

An *upper bound* of a subset  $B \subseteq A$  of a poset  $(A, \leq)$  is an element  $a \in A$  such that for all  $b \in B$  we have  $b \leq a$ .

A *least upper bound* (LUB) or *join* of  $B$  is an upper bound  $a$  such that for all other upper bounds  $a'$  we have  $a \leq a'$ .

The *join* of  $B$  is written  $\vee B$ .

When the join of  $B$  exists, then  $B$  is said to be *joinable*.

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## Meet (Greatest Lower Bound)

A *lower bound* of a subset  $B \subseteq A$  of a poset  $(A, \leq)$  is an element  $a \in A$  such that for all  $b \in B$  we have  $a \leq b$ .

A *greatest lower bound* (GLB) or *meet* of  $B$  is a lower bound  $a$  such that for all other lower bounds  $a'$  we have  $a' \leq a$ .

The *meet* of  $B$  is written  $\wedge B$ .

When the meet of  $B$  exists and is in  $B$ , then  $B$  is said to be *well-founded*. In this case, we call  $\wedge B$  the “bottom” of  $B$  and often write it  $\perp$ .

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## Example of Join and Meet

Example: Given a set  $A$  and its *powerset* (set of all subsets)  $P(A)$ , then  $(P(A), \subseteq)$  is a poset. For any  $B \subseteq P(A)$ , we have

$$\begin{aligned}\vee B &= \cup B \text{ (the union of the subsets) and} \\ \wedge B &= \cap B \text{ (the intersection of the subsets)}\end{aligned}$$

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## Complete Partial Order

A *complete partial order* (CPO) is a well-founded partially ordered set where every chain is joinable.

Example:  $(\mathbb{N}, \leq)$  is not a CPO.

Example:  $(\mathbb{N} \cup \{\infty\}, \leq)$  is a CPO.

Example:  $(A^{**}, \subseteq)$  is a CPO.

- The bottom element is the empty sequence.
- The join of any infinite chain is an infinite sequence.

Example:  $(A^*, \subseteq)$  is not a CPO.

- $A^*$  is the set of all finite sequences.

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## Monotonic (Order Preserving) Functions

Let  $(A, \leq)$  and  $(B, \leq)$  be posets.

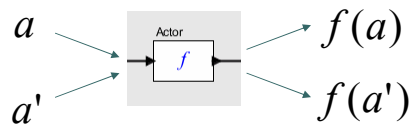
A function  $f: A \rightarrow B$  is called *monotonic* if

$$a \leq a' \Rightarrow f(a) \leq f(a')$$

Example: PN actors are monotonic with the prefix order.

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## PN Actors are Monotonic Functions on a CPO



Set of signals with the prefix order is a CPO.

Actors are *monotonic* functions:

$$a \sqsubseteq a' \Rightarrow f(a) \sqsubseteq f(a')$$

This is a timeless *causality* condition.

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## Example of a Non-Monotonic but Functional Actor

Unfair merge  $f: A \times A \rightarrow A$  where  $(A, \sqsubseteq)$  is a poset

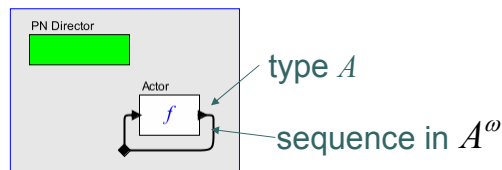
$$f(a,b) = \begin{cases} a & \text{if } a \text{ is infinite} \\ a.b & \text{otherwise} \end{cases}$$

where the period indicates concatenation.

Exercise: show that this function is not monotonic under the prefix order.

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## Fixed Point Semantics



- Start with the empty sequence.
- Apply the (monotonic) function.
- Apply the function again to the result.
- Repeat forever.

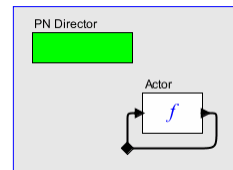
The result “converges” to the least fixed point.

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## Fixed Point Theorem 2

Let  $f: A \rightarrow A$  be a monotonic function on CPO  $A$ .  
Then  $f$  has a least fixed point.

Take the “meaning” or “semantics” of this process network to be that the (one and only) signal in the system is the least fixed point of  $f$ .



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## Conclusion

PN actors that are “causal” are monotonic functions on the CPO of sequences with the prefix order.

The semantics of a PN model with an actor feeding its own output back to its input is the least fixed point of the actor function.

Next time: Give a procedure for finding the fixed point and generalize to arbitrary process networks.

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