Concurrent Models of Computation for Embedded Software

Edward A. Lee

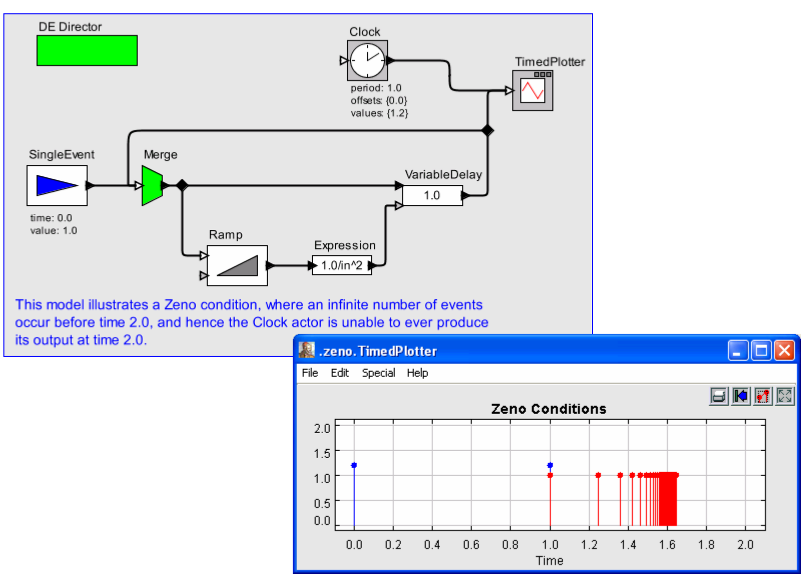
Professor, UC Berkeley

EECS 290n – Advanced Topics in Systems Theory

Fall, 2004

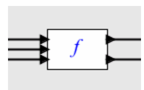
Lecture 13: Metric Space Semantics

We Seek Semantics that Give Meaning to Feedback and Help Rule Out Zeno



Mathematical Framework

Let the set of all signals be $A = [T \to V]$ where $T = R \times N$ is a totally ordered tag set and V is a set of values. Let an actor



be a function $f: A^n \to A^m$. What are the constraints on these functions such that:

1. Compositions of actors are determinate.
2. Feedback compositions have a meaning.
3. We can rule out Zeno behavior.

Metric

A *metric* on a set A is a function $d: A \times A \to R$ where for all $a, b, c \in A$

1. $d(a, b) = d(b, a)$
2. $d(a, b) = 0 \Leftrightarrow a = b$
3. $d(a, b) + d(b, c) \geq d(a, c)$

Exercise: Show that these properties imply that for all $a, b \in A, \quad d(a, b) \geq 0$

Metric space: (A, d)

Variations on Metrics

Ultrametric: Replace property 3 with:

3. $\max(d(a,b), d(b,c)) \geq d(a,c)$

Exercise: Prove that an ultrametric is a metric.

Partial Metric: Replace properties 2 and 3 with:

2. $d(a,a) \leq d(a,b)$
3. $d(a,b) + d(b,c) - d(b,b) \geq d(a,c)$

In a partial metric, a is the "closest" object to itself.

The Cantor Metric

Given the tag set $T = R \times N$ use only the time stamps. Let

$$d : [T \to V] \times [T \to V] \to R$$

such that for all $s, s' \in [T \to V]$,

$$d(s, s') = 1/2^{\tau}$$

where τ is the time stamp of the least tag t where $s(t) \neq s'(t)$. That is, either one is defined and the other not at t or both are defined but are not equal.

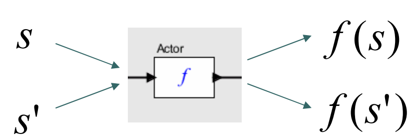
The Cantor Metric is an Ultrametric

Need to show that for all signals $a, b, c \in [T \to V]$,

1. $d(a, b) = d(b, a)$
2. $d(a, b) = 0 \Leftrightarrow a = b$
3. $\max(d(a, b), d(b, c)) \geq d(a, c)$

(1) and (2) are obvious. To show (3), assume without loss of generality that $d(a, b) \geq d(b, c)$. This means that a and b differ earlier than b and c. Suppose that a and b differ first at time τ. Since a and b differ earlier than b and c, then prior to τ, b and c are identical. Thus, a and c must be identical prior to τ so $d(a, c)$ must be smaller than or equal to $d(a, b)$. QED

Causality



Causal: For all signals s and s'

$$d(f(s), f(s')) \leq d(s, s')$$

Strictly causal: For all signals s and s'

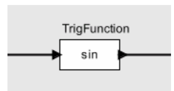
$$s \neq s' \Rightarrow d(f(s), f(s')) < d(s, s')$$

Delta causal: There exists a real number $\delta < 1$ such that for all signals s and s'

$$s \neq s' \Rightarrow d(f(s), f(s')) \leq \delta d(s, s')$$

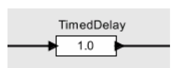
Examples

Simple functional actor:



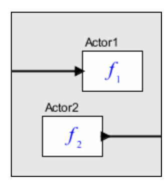
This actor is causal but not strictly causal or delta causal.

Time delay with non-zero delay:



This actor is delta causal.

Source and Sink Actors

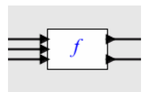


Consider Actor1. Its function is $f_1 \colon A^1 \to A^0$ where A^0 is a *singleton set* (a set with one element). Such a function is always delta causal with $\delta = 0$.

Consider Actor2. Its function is $f_1 \colon A^0 \to A^1$. Such a function is again always delta causal with $\delta = 0$.
In fact, the function can only yield one possible output signal, since its domain has size 1.

Extending to Multiple Inputs/Outputs

Consider a function $f : A^n \to A^m$, where $A = [T \to V]$



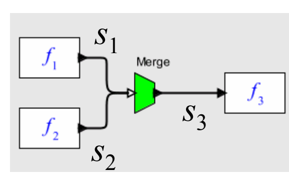
The input is a tuple of signals $(a_1, a_2, \dots, a_n)$.

Extend the Cantor metric to handle tuples:

$$d((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n))$$
$$= \min(d(a_1, b_1), \dots, d(a_n, b_n))$$

The resulting function is still an ultrametric.

Example: Merge Actor



Recall that for input

$$s_1 = \{\dots ((t, 0), v_1), ((t, 1), v_2), \dots)$$
$$s_2 = \{\dots ((t, 0), q_1), ((t, 1), q_2), \dots)$$

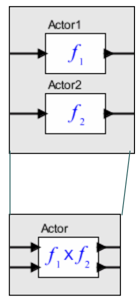
the output is:

$$s_3 = \{\dots ((t, 0), v_1), ((t, 1), q_1), ((t, 2), v_2), ((t, 3), q_2), \dots)$$

This actor is causal but not strictly causal, and the operations on indexes do not appear in the semantics.

Parallel Composition of Actors

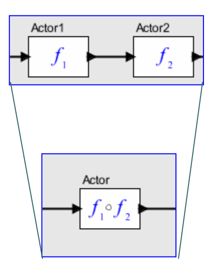
If f_1 and f_2 are causal (strictly causal, delta causal), then so is $f_1 \times f_2$.



What if f_1 is causal and f_2 is delta causal?

Cascade Composition of Actors

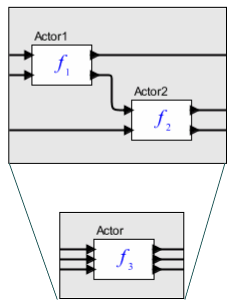
If f_1 and f_2 are causal (strictly causal, delta causal), then so is $f_1 \circ f_2$.



What if f_1 is causal and f_2 is delta causal?

More Interesting Composition

If f_1 and f_2 are causal (strictly causal, delta causal), then so is the following composition:



Question: What if f_1 is causal and f_2 is delta causal?

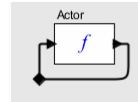
Technicality

In the set $S = [T \to V]$, we could have a signal s that has, for example, an event at all integer time stamps (positive and negative), and we could compare it against a signal s' that has no events at all.

$$d(s, s') = \infty$$

This is problematic. We can avoid these problems by excluding from the set S all signals that have infinite distance from the empty signal. All such signals have an earliest event.

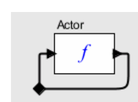
Feedback: Fixed Point Semantics



Since monotonicity on the prefix order is not very useful, we can't use fixed-point theorem 1.

Use instead fixed-point theorems on metric spaces.

Fixed Point Theorem 3



Let $(S^n = [T \to V]^n, d)$ be a metric space and $f : S^n \to S^n$ be a strictly causal function. Then f has at most one fixed point.

Proof: It is enough to show that

$$s \neq s' \;\Rightarrow\; f(s) \neq s \text{ or } f(s') \neq s'.$$

Suppose to the contrary that

$$s \neq s' \text{ and } f(s) = s \text{ and } f(s') = s'$$

But this is not possible because it would imply that

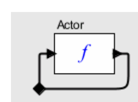
$$d(s, s') = d(f(s), f(s')) < d(s, s').$$

Determinacy

Fixed-Point Theorem 3 takes care of determinacy. There can be no more than one behavior.

Can we find that behavior?

Fixed Point Theorem 4 (Banach Fixed Point Theorem)



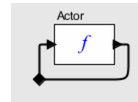
Let $(S^n = [T \to V]^n, d)$ be a *complete* metric space and $f: S^n \to S^n$ be a delta causal function. Then f has a unique fixed point, and for any point $s \in S^n$, the following sequence converges to that fixed point:

$$s_1 = s,\ s_2 = f(s_1),\ s_3 = f(s_2),\ \ldots$$

This means no Zeno! Two issues:

- Any starting point?
- Complete metric space?

Construction of a Fixed Point: Example



Suppose f is a delay by one time unit, such that

$$s' = f(s)$$

where for each event $e = (t, v) \in s$ where $t = (\tau, n)$, there is an event $e' = (t', v) \in s'$ where $t' = (\tau + 1, n)$.

Suppose we start with a "lucky guess" $s = \emptyset$. This is the only fixed point, so we converge immediately.

Suppose we start with an "unlucky guess" $s = \{((0,0), 0)\}$. As we iterate f, the event gets further out in the future, and the signal "converges" to $s = \emptyset$.

Complete Metric Spaces

A *Cauchy sequence* $\{s_1, s_2, \ldots\}$ is an infinite sequence where

$$d(s_n, s_m) \to 0 \text{ as } n, m \to \infty$$

A *complete metric space* (X, d) is one where every Cauchy sequence has a limit in X.

Example 1

Consider a sequence $\{s_1, s_2, \ldots\}$ where

$$s_n = \{((n, 0), v)\}$$

Is this sequence Cauchy?

Does the sequence converge? To what?

Example 1

Consider a sequence $\{s_1, s_2, \ldots\}$ where

$$s_n = \{((n, 0), v)\}$$

Is this sequence Cauchy? Yes

$$d(s_n, s_m) = 1/2^{\min(m, n)} \to 0$$

Does the sequence converge? To what? Yes. To $\varnothing$

$$\lim (s_n) = \varnothing$$

Example 2

Consider a sequence $\{s_1, s_2, \ldots\}$ where

$$s_n = \{((i, 0), v) \mid i \in \{1, 2, \ldots, n\}\}$$

Is this sequence Cauchy?

Does the sequence converge? To what?

Example 2

Consider a sequence $\{s_1, s_2, \ldots\}$ where

$$s_n = \{((i, 0), v) \mid i \in \{1, 2, \ldots, n\}\}$$

Is this sequence Cauchy? Yes

$$d(s_n, s_m) = 1/2^{\min(m, n) + 1} \to 0$$

Does the sequence converge? To what? Yes. To

$$\{((i, 0), v) \mid i \in \{1, 2, \ldots\}\}$$

Example 3

Consider a sequence $\{s_1, s_2, \ldots\}$ where

$$s_n = \{((\tau_i, 0), v) \mid i \in \{1, 2, \ldots, n\}, \tau_i = 1 - 1/i\}$$

Is this sequence Cauchy?

Does the sequence converge? To what?

Example 3

Consider a sequence $\{s_1, s_2, \ldots\}$ where

$$s_n = \{((\tau_i, 0), v) \mid i \in \{1, 2, \ldots, n\}, \tau_i = 1 - 1/i\}$$

Is this sequence Cauchy? No

$$d(s_n, s_m) > 1/2$$

Does the sequence converge? To what? No. Exercise.

Completeness of DE Signals

The set of n-tuples of discrete-event signals under the Cantor metric is a complete metric space.

Proof (sketch): We need to show that every Cauchy sequence converges. Given a Cauchy sequence $\{s_1, s_2, \ldots\}$, for any tag t with time stamp $\tau > 0$, there is a subsequence $\{s_n, s_{n+1}, \ldots\}$, for some $n > 0$, of signals that are identical up to and including tag t. Let s be the sequence obtained by letting its value at each tag t be that identical value (or absence, if all signals in the subsequence have no event at t). This is clearly a signal (or tuple of signals). Then it is easy to show that the Cauchy sequence converges to s.

Thanks to Adam Cataldo for this proof.

Operational Semantics

1. Topologically sort actors according to paths that do not increment tags.
2. Start with a set of events on signals taken from the event queue that all have the same tag.
3. Iterate to find a fixed-point value for all signals at that tag (absent or having a value).
4. Continue with the next smallest tag in the event queue.

Conclusions and Open Issues

- Ignoring the index, strictly causal functions in a feedback loop have at most one fixed point, and hence are determinate.
- Delta causal functions in a feedback loop have exactly one fixed point, and that fixed point can be found by starting with any initial signal(s) and iterating to the fixed point. This guarantees no Zeno.
- Convergence in DE is achieved when time stamps approach infinity.
- Within a time stamp, use SR semantics and iterate to a fixed point.