Linear-Time vs. Branching-Time Properties

So far we have been talking about properties of linear behaviors (sequences, traces).

But some properties are not linear, e.g.:

“it is possible to recover from any fault”

or

“there exists a way to get back to the initial state from any reachable state”
Linear-Time vs. Branching-Time Properties

“it is possible to recover from any fault”

Based on one (linear) behavior alone,\(^1\) we cannot conclude whether our system satisfies the property.

E.g., the following system satisfies the property, although it contains a behavior that stays forever in state \(s_1\):

\[
\begin{array}{c}
\text{fault} \\
\rightarrow \\
\text{recovery}
\end{array}
\]

\(s_0\) \(\rightarrow\) \(s_1\)

---

\(^1\)if we had all linear behaviors of a system, we could in principle reconstruct its branching behavior as well

Linear-Time vs. Branching-Time Temporal Logics

**Linear-time**: the “solutions” (models) of a temporal logic formula are infinite **sequences** (traces).

**Branching-time**: the “solutions” (models) of a temporal logic formula are infinite **trees**.

- Hence the name “Computation Tree Logic” for CTL.
We will simplify and define the semantics of CTL directly on states of a transition system (Kripke structure).

CTL (Computation Tree Logic) – Syntax

CTL formulas are defined by the following grammar:

\[
\phi ::= p \mid q \mid \ldots \\
| \phi_1 \land \phi_2 \mid \neg \phi_1 \\
| \text{EG}\phi_1 \mid \text{AG}\phi_1 \\
| \text{EF}\phi_1 \mid \text{AF}\phi_1 \\
| \text{EX}\phi_1 \mid \text{AX}\phi_1 \\
| E(\phi_1 U \phi_2) \mid A(\phi_1 U \phi_2)
\]

E (“there exists a path”) and A (“for all paths”) are called path quantifiers.
CTL (Computation Tree Logic) – Syntax

Examples of CTL formulas:

\[ \text{AG} p \]
\[ \text{EF} q \]
\[ \text{AGEF}(p \Rightarrow q) \]

Alternative notation: \( \forall \square p, \exists \diamond q, \forall (p \cup q) \), etc.

CTL – Semantics: Intuition

Let \( s \) be a state of the Kripke structure. Then \( s \) satisfies the CTL formula \( \text{EG} \phi \), written

\[ s \models \text{EG} \phi \]

iff there exists a trace \( \sigma \) starting from \( s \) and satisfying \( \text{G} \phi \), i.e.:

\[ \exists \text{ trace } \sigma \text{ starting from } s : \sigma \models \text{G} \phi \]

Note:

- The 2nd \( \models \) is the LTL satisfaction relation.
- The 1st \( \models \) refers to the CTL satisfaction relation.
- To be more pedantic:

\[ s \models_{\text{CTL}} \text{EG} \phi \quad \text{iff} \quad \exists \text{ trace } \sigma \text{ starting from } s \text{ s.t. } \sigma \models_{\text{LTL}} \text{G} \phi \]
CTL – Semantics: Intuition

\[ s \models \text{AG}\phi \]

iff every trace \( \sigma \) starting from \( s \) satisfies \( \text{G}\phi \):

\[ \forall \text{ traces } \sigma \text{ starting from } s : \sigma \models \text{G}\phi \]

Examples

Let’s construct transition systems (Kripke structures) satisfying or violating the following CTL formulas:

\[ \text{AG}p \]

\[ \text{AF}p \]

\[ \text{EG}p \]

\[ \text{EF}p \]
Facts about CTL

Quiz: do we need EF$\phi$? Can we express it in terms of other CTL modalities?

CTL – Formal Semantics

Let $(P, S, S_0, L, R)$ be a Kripke structure and let $s \in S$.

A trace starting from $s$ is an infinite sequence $\sigma = \sigma_0, \sigma_1, \cdots$, such that there is an infinite path $s = s_0, s_1, \cdots$ starting from $s$, and $\sigma_i = L(s_i)$ for all $i$.

Satisfaction relation for CTL:

\begin{align*}
  s \models p & \quad \text{iff} \quad p \in L(s) \\
  s \models \phi_1 \land \phi_2 & \quad \text{iff} \quad s \models \phi_1 \text{ and } s \models \phi_2 \\
  s \models \neg \phi & \quad \text{iff} \quad s \not\models \phi \\
  s \models \text{EG} \phi & \quad \text{iff} \quad \exists \text{trace } \sigma \text{ starting from } s : \sigma \models \text{LTL} \ \text{G} \phi \\
  s \models \text{AG} \phi & \quad \text{iff} \quad \forall \text{traces } \sigma \text{ starting from } s : s \models \text{LTL} \ \text{G} \phi \\
  s \models \text{EX} \phi & \quad \text{iff} \quad \exists \text{trace } \sigma \text{ starting from } s : s \models \text{LTL} \ \text{X} \phi \\
  s \models E(\phi_1 \ U \phi_2) & \quad \text{iff} \quad \exists \text{trace } \sigma \text{ starting from } s : s \models \text{LTL} \ \phi_1 \ U \phi_2 \\
  \ldots
\end{align*}
CTL – Examples

How to express these properties in CTL?

“\( p \) holds at all reachable states” \( \text{AG}\ p \)

“there exists a way to get back to the initial state from any reachable state” \( \text{AG EF init} \)

“\( p \) is inevitable” \( \text{AF} \ p \)

“\( p \) is possible” \( \text{EF} \ p \)

How would you express the last two in LTL?

Bibliography