Timed Automata

Stavros Tripakis
University of California, Berkeley
Timed Automata

- A formal model for dense-time systems [Alur and Dill(1994)]
- Developed mainly with verification in mind:
  - in the basic TA variant, model-checking is decidable
- But also an elegant theoretical extension of the standard theory of regular and \( \omega \)-regular languages.
- Many different TA variants, some undecidable.
- We will look at a basic variant.
Timed Automaton

A TA is a tuple

\[(C, Q, q_0, \text{Inv}, \triangleright)\]

- \(C\): finite set of *clocks*
- \(Q\): finite set of *control states*; \(q_0 \in Q\): initial control state
- \(\text{Inv}\): a function assigning to each \(q \in Q\) an invariant
- \(\triangleright\): a finite set of *actions*, each being a tuple
  \[(q, q', g, C')\]
  - \(q, q' \in Q\): source and destination control states
  - \(g\): clock *guard*
  - \(C'\): set of clocks to *reset* to 0, \(C' \subseteq C\)

- Invariants and guards are simple constraints on clocks, e.g.,
  \[c \leq 1, \quad 0 < c_1 < 2 \land c_2 = 4, \quad \text{etc.}\]
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Can also have atomic propositions labeling control states, labels on actions, communication via shared memory or message passing, etc.
Example: Timed Automaton

A simple light controller:

- \( C = \{ c \} \)
- \( Q = \{ \text{off, light, bright} \} \)
- \( q_0 = \text{off} \)
- touch: action label (can be seen as the input symbol)
- \( \text{Inv}(q) = \text{true} \) for all \( q \in Q \)
- Actions: (off, light, \text{true}, \{ c \}), (light, off, c \geq 2, \{ \}), ...
Event-based vs. state-based models

High-level:

An over-voltage threshold of 120 is exceeded during startup, and with this controller, the fault condition disables connecting the load.

Specification here is not met by this implementation, so if the connection is made to the ThrowModelError2 actor, then an exception will result on running the model.

Low-level:

See also a cleaner version of this model, where the specification monitor is an aspect.
Timed Automata: Semantics

A TA \((C, Q, q_0, \text{Inv}, \triangleright)\) defines a transition system

\[(S, S_0, R)\]

such that

- **Set of states:** \(S = Q \times \mathbb{R}_+^C\)
  - \(\mathbb{R}_+^C\): the set of all functions \(v : C \to \mathbb{R}_+\)
  - each \(v\) is called a *valuation*: it assigns a value to every clock

- **Set of initial states:** \(S_0 = \{(q_0, v_0)\}\), where we define \(v_0(c) = 0\) for all \(c \in C\) (i.e., all clocks are initially set to 0)

- **Set of transitions:** \(R = R_t \cup R_d\)
  - \(R_t\): set of transitions modeling passage of time
  - \(R_d\): set of discrete transitions (“jumps” between control states)
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  - we could also define \(S_0 = \{q_0\} \times \mathbb{R}_+^C\) – what does this say?

- **Set of transitions:** \(R = R_t \cup R_d\)
  - \(R_t\): set of transitions modeling passage of time
  - \(R_d\): set of discrete transitions (“jumps” between control states)
\[ R_t = \{(q, v), (q, v + t)\} | \forall t' \leq t : v + t' \models \text{Inv}(q) \} \]

\[ R_d = \{(q, v), (q', v')\} | \exists a = (q, q', g, C') \in \triangledown : \\
\text{v} \models g \land v' = v[C' := 0] \} \]

where:

- \( v + t \) is a new valuation \( u \) such that \( u(c) = v(c) + t \) for all \( c \)
- if \( g \) is a constraint, then \( v \models g \) means \( v \) satisfies \( g \)
- \( v[C' := 0] \) is a new valuation \( u \) such that \( u(c) = 0 \) if \( c \in C' \) and \( u(c) = v(c) \) otherwise
Timed Automata: Discrete and Time Transitions

\[ R_t = \{ ((q, v), (q, v + t)) \mid \forall t' \leq t : v + t' \models \text{Inv}(q) \} \]
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Instead of \( ((q, v), (q, v + t)) \in R_t \) we write \( (q, v) \xrightarrow{t} (q, v + t) \).
Instead of \( ((q, v), (q', v')) \in R_d \) we write \( (q, v) \xrightarrow{a} (q', v') \).
Example: Alarm Modeled as a Timed Automaton

\[ \text{Inv}(\text{off}) = c \leq 10 : \text{automaton cannot spend more than 10 time units at control state “off”}. \]
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What if we omit the invariant?
Example: Alarm Modeled as a Timed Automaton

Does it work correctly if cancel arrives exactly when $c = 10$?
Example: Alarm Modeled as a Timed Automaton

Does it work correctly if cancel arrives exactly when \( c = 10 \)?

Depends on the semantics of composition: if it’s non-deterministic (as usually done) then alarm may still ring. Otherwise, must give higher priority to the cancel transition.
Basic question: is a given control state \( q \) reachable?

- i.e., does there exist some reachable state \( s = (q, v) \) in the transition system defined by the timed automaton?

Many interesting questions about timed automata can be reduced to this question.
Basic question: is a given *control state* $q$ *reachable*?

- i.e., does there exist some reachable state $s = (q, v)$ in the transition system defined by the timed automaton?

Many interesting questions about timed automata can be reduced to this question.

Is the basic control-state reachability question decidable?
Not the same as discrete-state reachability!

$q_4$ is reachable if we ignore the timing constraints. But is it really reachable?
Timed Automata Reachability

Not the same as discrete-state reachability!

$q_4$ is reachable if we ignore the timing constraints. But is it really reachable?

No: at $q_3$, $c_2 > 1$ and $c_1 \geq c_2$, therefore $c_1 > 1$ also.
A less obvious example: Fischer’s mutual exclusion protocol.

Suppose we have many processes, each behaving like the TA above. Is mutual-exclusion guaranteed? I.e., at most 1 process is in critical section (control state $cs$) at any given time.
Timed Automata Model-Checking: Reachability

Brute-force idea: exhaustive state-space exploration of the transition system defined by the timed automaton

- does not work since state-space is infinite (even uncountable)
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- does not work since state-space is infinite (even uncountable)

Yet problem is decidable! [Alur-Dill’94]

Key idea:

- Region equivalence: partitions the state-space into finite number of equivalence classes (regions)
- Perform reachability on finite (abstract) state-space
- Can prove that \( q \) is reachable in the abstract space iff it is reachable in the concrete space
The Region Equivalence

Key idea: two valuations \(v_1, v_2\) are equivalent iff:

1. \(v_1\) satisfies a guard \(g\) iff \(v_2\) satisfies \(g\).
2. \(v_1\) can lead to some \(v'_1\) satisfying a guard \(g\) with a discrete transition iff \(v_2\) can do the same.
3. \(v_1\) can lead to some \(v'_1\) satisfying a guard \(g\) with a time transition iff \(v_2\) can do the same.

Region = equivalence class w.r.t. region equivalence = set of all equivalent valuations.

Region in gray:
\[1 < x < 2 \land 1 < y < 2 \land x > y.\]

Other regions:
\[x = y = 0,\]
\[0 < x = y < 1,\]
\[x = 0 \land 0 < y < 1,\]

etc.

Pictures in this and other slides taken from [Bouyer(2005)].
The Region Equivalence: Finiteness

Finite number of equivalence classes: bounded by constant $c = \text{maximal constant appearing in a guard or invariant}$.

Some regions are unbounded, e.g.:

- $x > 2 \land 0 < y < 1$
- $x > 2 \land y > 2$
- etc.
The Region Graph

A graph of regions: one region space for each control location.

Nodes: pairs \((q, r)\) where
- \(q\) is a control location of the timed automaton.
- \(r\) is a region.

Two types of edges:
- \((q, r) \xrightarrow{a} (q', r')\): discrete transition
- \((q, r) \xrightarrow{\text{time}} (q, r')\): time transition
Decidability

Theorem ([Alur and Dill(1994)])

\[ \exists \text{ reachable state } (q, v) \text{ in a timed automaton } \]
\[ \iff \]
\[ \exists \text{ reachable node } (q, r) \text{ in its region graph. } \]

Finite \# regions and control states \Rightarrow Region graph is finite \Rightarrow Reachability is decidable.
The Problem with Regions

STATE EXPLOSION!

Worst-case number of regions:

\[ O(2^n \cdot n! \cdot c^n) \]

where \( n \) is the number of clocks and \( c \) is the maximal constant.

This is actually often close to the actual number of regions \( \Rightarrow \) no practical tool uses regions.

Model-checkers for TA (Uppaal, Kronos, ...) have improved upon the region-graph idea and use symbolic techniques.
From Regions to Zones

Zone: a convex union of regions, e.g., \( x_1 \geq 3 \land x_2 \leq 5 \land x_1 - x_2 \leq 4 \).
Key property: can be represented efficiently using difference bound matrices (DBMs) [Dill(1989)].

\[ x_1 \geq 3 \land x_2 \leq 5 \land x_1 \leq x_2 + 4 \]

\[
\begin{pmatrix}
\infty & -3 & \infty \\
\infty & \infty & 4 \\
5 & \infty & \infty
\end{pmatrix}
\]
Symbolic Manipulations of Zones using DBMs

DBMs = the BDDs of the timed automata world.

Time elapse, guard intersection, clock resets, are all easily implementable in DBMs.
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Is zone union implementable with DBMs?
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DBMs = the BDDs of the timed automata world.

Time elapse, guard intersection, clock resets, are all easily implementable in DBMs.

Is zone union implementable with DBMs?

No! The union of two zones in general is not a zone.

⇒ often state explosion even with zones ...
Timed Automata Reachability: Simple Example

Is $q_2$ reachable? (initially, $c_1 = c_2 = 0$)
Timed Automata Reachability: Simple Example

Is $q_2$ reachable? (initially, $c_1 = c_2 = 0$)

Symbolic reachability analysis:

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**Timed Automata Reachability: Simple Example**

![Diagram](image)

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Therefore, $q_2$ is not reachable.
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therefore $q_2$ not reachable
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