Homework 4

Due 11/07/00

• Exercise 1
Let X be a non-empty set and d a function $d : X \times X \rightarrow \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers), such that $\forall x, y, z \in X$:

1. $d(x, y) = d(y, x)$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) \geq 0$
4. $d(x, y) + d(y, z) \geq d(x, z)$

Then, d is called a metric on X and $(X, d)$ is a metric space.

a) Define a metric d on $X = \mathbb{R}$.

b) Prove that the Cantor metric defined on the set $S^n$ of n-tuples of signals (signals and tags are defined in the paper A Denotational Framework for Comparing Models of Computation by Lee and Sangiovanni-Vincenelli) as

$$d(s, s') = \sup \left\{ \frac{1}{2^t} : s(t) \neq s'(t), t \in [0, \infty) \right\}$$

is an ultrametric (d is an ultrametric, if satisfies also $\max(d(s, s'), d(s', s'')) \geq d(s, s'')$ in addition to (1)(2)(3)(4)).

• Exercise 2
Let P and Q be ordered sets. A map $\varphi : P \rightarrow Q$ is said to be:

- order-preserving (or monotonic) if $x \leq y$ in P implies $\varphi(x) \leq \varphi(y)$ in Q
- order-embedding if $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q
- order-isomorphism if it is an order-embedding mapping P onto Q (A map $f : P \rightarrow Q$ is onto (or surjective) if for every $y \in Q$, there exists an element $x \in P$, such that $f(x) = y$).

a) Let $\mathbb{N}$ be the set of natural numbers and $N_0 = \mathbb{N} \cup \{0\}$. The partial order $\preceq$ on $N_0$ is defined as follows: $\forall m, n \in N_0, m \preceq n$ if and only if $\exists k \in N_0$ such that $km = n$ (e.g. $3 \preceq 6, 4 \preceq 12$).
Consider the ordered sets \( P \) and \( Q \) s.t. \( P = Q = (N_0, \leq) \), is the map \( \varphi : P \rightarrow Q \) defined as \( \varphi(x) = 2x \) monotonic? Is \( \varphi(x) = x + 3 \) monotonic?

b) Let \( \varphi(N) \) be the powerset of \( N \), consisting of all subsets of \( N \) and \( \leq \) the partial order on \( \varphi(N) \) defined as: \( \forall A, B \in \varphi(N), A \leq B \) if and only if \( A \subseteq B \).

Consider the ordered sets \( P \) and \( Q \) s.t. \( P = Q = (\varphi(N), \leq) \), is the map \( \varphi \) defined as:

\[
\varphi(U) = \begin{cases} 
1 & \text{if } 1 \in U, \\
2 & \text{if } 2 \in U \text{ and } 1 \notin U, \\
0 & \text{otherwise}.
\end{cases}
\]

monotonic?

c) Let \( X = \{1, 2, \ldots, n\} \). Consider two ordered sets \((\varphi(X), \subseteq)\) and \((P^n, \leq)\), where \( P = \{0, 1\} \) and \( \leq \) is defined as \( (x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \) if and only if \( \forall i \: x_i \leq y_i \) in \( P \).

Consider the map \( \varphi : \varphi(X) \rightarrow 2^n \) defined as \( \varphi(A) = (e_1, e_2, \ldots, e_n) \) where

\[
e_i = \begin{cases} 
1 & \text{if } i \in A, \\
0 & \text{if } i \notin A.
\end{cases}
\]

Show that \( \varphi \) is an order-isomorphism.

- **Exercise 3** Let \( P \) be a set. A partial order on \( P \) is a binary relation \( \leq \) on \( P \) such that for all \( x, y, z \in P \):
  
  - \( x \leq x \)
  - if \( x \leq y \) and \( y \leq x \), then \( x = y \)
  - if \( x \leq y \) and \( y \leq z \), then \( x \leq z \)

An ordered set \( P \) is a **chain** if for all \( x, y \in P \), either \( x \leq y \) or \( y \leq x \).

a) Define \( \leq \) on \( N \) such that \((N, \leq)\) is a chain.

b) Define \( \leq \) on \( N \) such that \((N, \leq)\) is not a chain.

c) Let \( P \) and \( Q \) be ordered sets and \( P \times Q \) their cartesian product. Is the binary relation \( \leq \) defined as

\[
(x_1, x_2) \leq (y_1, y_2) \text{ if } x_1 < y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 \leq y_2)
\]

a partial order on \( P \times Q \)?
Exercise 4

Using the Tagged Signal Model (defined in the paper *A Denotational Framework for Comparing Models of Computation* by Lee and Sangiovanni-Vincentelli), model the filter

\[ o(n) = k_1 \ i(n) + k_2 \ o(n - 1) \]

where \( n \) is the index of the samples, \( k_1 \) and \( k_2 \) are given coefficients. Assume that an initial token (event) \( o(0) \) is present and is used by the \( k_2 \) multiplier in the first iteration.

Define all the processes in Figure 1 and the composite process of all the processes and connections.

![Figure 1: Filter \( o(n) = k_1 \ i(n) + k_2 \ o(n - 1) \)](image)