Concurrent Models of Computation

Edward A. Lee

Robert S. Pepper Distinguished Professor, UC Berkeley

EECS 290n – Advanced Topics in Systems Theory

Concurrent Models of Computation

Spring 2009

Week 2: Fixed-Point Semantics

Review

- Sets, tuples, relations, functions, powersets
- Partially ordered sets
- Chains
- Well ordered sets (chains where every non-empty subset has a least element)
- The set T^{**} of all finite and infinite sequences from T
- Prefix order

Join (Least Upper Bound)

An *upper bound* of a subset $B \subseteq A$ of a poset $(A, \leq)$ is an element $a \in A$ such that for all $b \in B$ we have $b \leq a$.

A *least upper bound* (LUB) or *join* of B is an upper bound a such that for all other upper bounds a' we have $a \leq a'$.

The *join* of B is written $\vee B$.

When the join of B exists in A, then B is said to be *joinable*.

Meet (Greatest Lower Bound)

A *lower bound* of a subset $B \subseteq A$ of a poset $(A, \leq)$ is an element $a \in A$ such that for all $b \in B$ we have $a \leq b$.

A *greatest lower bound* (GLB) or *meet* of B is a lower bound a such that for all other lower bounds a' we have $a' \leq a$.

The *meet* of B is written $\wedge B$.

Example of Join and Meet

Example: Given a set A and its *powerset* (set of all subsets) $P(A)$, then $(P(A), \subseteq)$ is a poset. For any $B \subseteq P(A)$, we have

$\vee B = \cup B$ (the union of the subsets) and

$\wedge B = \cap B$ (the intersection of the subsets)

Directed Sets, Bottom

A subset $B \subseteq A$ of a poset $(A, \leq)$ is *directed* if every pair of elements in B has an upper bound in A.

Equivalently, B is *directed* if every finite subset of B is joinable.

A poset $(A, \leq)$ has a *bottom* element if $\wedge A$ exists. We write it $\perp = \wedge A$.

Complete Partial Order

A *complete partial order* (CPO) is a partially ordered set with a bottom where every directed set is joinable.

Example: Every *finite* poset with a bottom is a CPO.

Example: $(N, \leq)$ is not a CPO.

Example: $(N \cup \{\infty\}, \leq)$ is a CPO.

Example: $(T^{**}, \sqsubseteq)$ is a CPO.

- The bottom element is the empty sequence.
- Every directed set is chain
- The join of any infinite chain is an infinite sequence.

Example: $(T^{*}, \sqsubseteq)$ is not a CPO.

- T^{*} is the set of all finite sequences.

Equivalent definition: Complete Partial Order

A poset A is a CPO iff every chain has a least upper bound in A.

The equivalence is nontrivial. See Davey & Priestley, theorem 8.11.

The equivalence is trivial for $A = T^{**}$, because every directed set in T^{**} is a chain.

Monotonic (Order Preserving) Functions

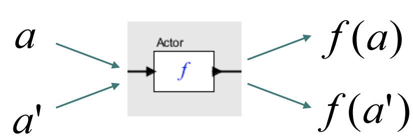
Let $(A, \leq)$ and $(B, \leq)$ be posets.

A function $f : A \to B$ is called *monotonic* if

$$a \leq a' \quad \Rightarrow \quad f(a) \leq f(a')$$

Example: PN actors are monotonic with the prefix order.

PN Actors are Monotonic Functions on a CPO



Set of signals with the prefix order is a CPO.

Actors are *monotonic* functions:

$$a \sqsubseteq a' \quad \Rightarrow \quad f(a) \sqsubseteq f(a')$$

This is a timeless *causality* condition.

Example of a Non-Monotonic but Functional Actor

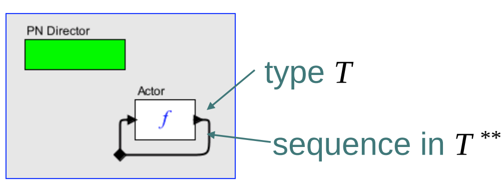
Unfair merge $f : A \times A \to A$ where $(A, \sqsubseteq)$ is a poset

$$f(a,b) = \begin{cases} a & \text{if } a \text{ is infinite} \\ a.b & \text{otherwise} \end{cases}$$

where the period indicates concatenation.

Exercise: show that this function is not monotonic under the prefix order.

Fixed Point Semantics



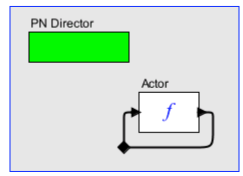
- Start with the empty sequence.
- Apply the (monotonic) function.
- Apply the function again to the result.
- Repeat forever.

The result “converges” to the least fixed point.

Fixed Point Theorem 2

Let $f: A \to A$ be a monotonic function on CPO A. Then f has a least fixed point.

Take the “meaning” or “semantics” of this process network to be that the (one and only) signal in the system is the least fixed point of f.



Continuous (Limit Preserving) Functions

Let $(A, \leq)$ and $(B, \leq)$ be CPOs.

A function $f: A \to B$ is called *continuous* if for all chains $C \subseteq A$,

$$f(\vee C) = \vee \hat{f}(C)$$

Notation: Given a function $f: A \to B$, define a new function $\hat{f}: P(A) \to P(B)$, where for any $C \subseteq A$,

$$\hat{f}(C) = \{b \in B \mid \exists c \in C \text{ s.t. } f(c) = b\}$$

Continuous vs. Monotonic

Fact: Every continuous function is monotonic.

- Easy to show (consider chains of length 2)

Fact: If every chain in A is finite, then every monotonic function $f : A \to B$ is continuous.

But: If A has infinite chains, the monotonic does not imply continuous.

Counterexample Showing that Monotonic Does Not Imply Continuous

Let $A = (N \cup \{\infty\}, \leq)$ (a CPO).
Let $f : A \to A$ be given by

$$f(a) = \begin{cases} 1 & \text{if } a \text{ is finite} \\ 2 & \text{otherwise} \end{cases}$$

This function is obviously monotonic. But it is not continuous. To see that, let $C = \{1, 2, 3, \ldots\}$, and note that $\vee C = \infty$. Hence,

$$f(\vee C) = 2$$
$$\vee f(C) = 1$$

which are not equal.

Intuition

Under the prefix order, for any monotonic function that is not continuous, there is a continuous function that yields the same result for every finite input.

For practical purposes, we can assume that any monotonic function is continuous, because the only exceptions will be functions that wait for infinite input before producing output.

Fixed Point Theorem 1

Let $(A, \leq)$ be a CPO
Let $f : A \to A$ be a monotonic function
Let $C = \{ f^n(\perp), n \in N \}$

- If $\vee C = f(\vee C)$, then $\vee C$ is the *least* fixed point of f
- If f is continuous, then $\vee C = f(\vee C)$

Intuition: The least fixed point of a continuous function is obtained by applying the function first to the empty sequence, then to the result, then to that result, etc.

Proof (Least Fixed Point Part)

NOTE: This part does not require continuity.

Let a be another fixed point: $f(a) = a$

Show that $\vee C$ is the least fixed point: $\vee C \leq a$

Since f is monotonic:

$\perp \leq a$

$f(\perp) \leq f(a) = a$

$\dots$

$f^k(\perp) \leq f^k(a) = a$

So a is an upper bound of the chain C, hence $\vee C \leq a$.

Lee 02: 19

Proof (Continuous Part)

Note that C is a chain in a CPO (show this) and hence has a LUB $\vee C$.

Let $C' = C \cup \{ \perp \}$ and note that $\vee C = \vee C'$.

Note further that $\hat{f}(C') = C$ and hence $\vee \hat{f}(C') = \vee C$

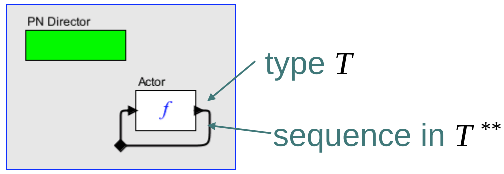
By continuity, $\vee \hat{f}(C') = f(\vee C') = f(\vee C)$

Hence $\vee C = f(\vee C)$

QED ($\vee C$ is a fixed point of f)

Lee 02: 20

Fixed Point Semantics



- Start with the empty sequence.
- Apply the (continuous) function.
- Apply the function again to the result.
- Repeat forever.

The result "converges" to the least fixed point.

Fixed Point Theorem 2

Let $f : A \to A$ be a monotonic function on CPO $(A, \leq)$.
Then f has a least fixed point.

Intuition: If a function is monotonic (but not continuous), then it has a least fixed point, but the execution procedure of starting with the empty sequence and iterating may not converge to that fixed point.

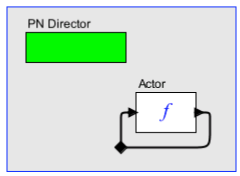
This is obvious, since monotonic but not continuous means it waits forever to produce output.

Example 1: Identity Function

Let $A = T^{**}$ and $f : A \to A$ be such that $\forall a \in A$, $f(a) = a$.

This is obviously continuous (and hence monotonic) under the prefix order.

Then the model below has many fixed points, but only one least fixed point (the empty sequence).

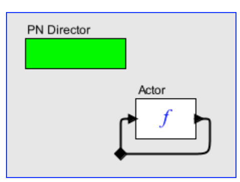


Example 2: Delay Function

Let $A = T^{**}$ and $f : A \to A$ be such that $\forall a \in A$, $f(a) = t.a$ (concatenation), where $t \in T$.

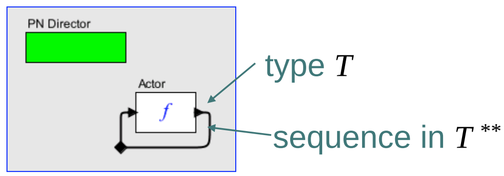
This is obviously continuous (and hence monotonic) under the prefix order.

Then the model below has only one fixed point, the infinite sequence $(t, t, t, \ldots)$



Why is this called a “delay?”

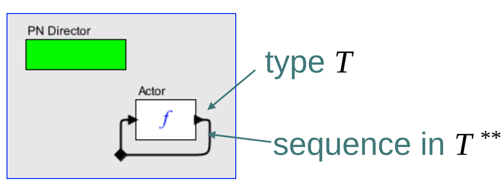
Taking Stock: Fixed Point Semantics



- Start with the empty sequence.
- Apply the (continuous) function.
- Apply the function again to the result.
- Repeat forever.

The result "converges" to the least fixed point.

Semantics of a PN Model is the Least Fixed Point of a Monotonic Function



- Chain: $C = \{ f(\perp), f(f(\perp)), \ldots, f^{n}(\perp), \ldots \}$
- Continuity: $f(\vee C) = \vee \hat{f}(C)$

Limits

Summary

- Posets
- CPOs
- Fixed-point theorems
- Gives meaning to simple programs
- With composition, gives meaning to all programs

- *Next time:*
  - *expressiveness of PN (Turing computability)*
  - *develop an execution policy*
  - *sequential functions, stable functions, and continuous functions*